Proceeding of the

7th Seminar on Reliability Theory and its Applications

University of Birjand, Iran

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Preface

Following the series of workshops on Reliability Theory and its Applications in Ferdowsi University of Mashhad and six seminars in University of Isfahan (2015), University of Tehran (2016), Ferdowsi University of Mashhad (2017), Shiraz University (2018), University of Yazd (2019) and University of Mazandaran (2020) we are pleased to organize the 7th seminar on **Reliability Theory and its Applications** during 19-20 May, 2021 at the Department of Statistics, University of Birjand. On behalf of the organizing and scientific committees, we would like to extend a very warm welcome to all participants and hope that this seminar provides a platform for useful discussions and would also exchange scientific ideas through interact of opinions. We wish to express our gratitude to the individuals for having any contribution to this seminar, particularly 60 colleagues, researchers, and postgraduate students from universities and organizations have participated.

Finally, we would like to extend our sincere gratitude to the Research Council of the University of Birjand, the administration of Faculty of Mathematical Sciences, the Ordered Data, Reliability and Dependency Center of Excellence, the Islamic World Science Citation Center, the Iranian Statistical Society, the Scientific Committee, the Organizing Committee, the referees, and the students and staff of the Department of Statistics at the University of Birjand for their kind cooperation.

Majid Chahkandi (Chairman)

July, 2021

Topics

The aim of the seminar is to provide a forum for presentation and discussion of scientific works covering theories and methods in the field of reliability and its application in a wide range of areas:

- Accelerated life testing
- Bayesian methods in reliability
- Case studies in reliability analysis
- Computational algorithms in reliability
- Data mining in reliability
- Degradation models
- Lifetime data analysis
- Lifetime distributions theory
- Maintenance modeling and analysis
- Networks reliability
- Optimization methods in reliability
- Reliability of coherent systems
- Safety and risk assessment
- Software reliability
- Stochastic aging
- Stochastic dependence in reliability
- Stochastic orderings in reliability
- Stochastic processes in reliability
- Stress-strength modeling
- Survival analysis

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Table of Contents

Several Bounds for Lifetime of Mixed Harris Family

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Abstract: As a lifetime distribution, Harris family distributions are applied as the lifetime of a series system with random number of components. In this paper, properties of various ageing classes of mixtures of Harris family distribution, where its tilt parameter is taken as a random variable, are studied. We obtain an upper bound for maximum error in evaluating the reliability function. Two bounds are also presented for survival function and expectation of mixed Harris family. We also provide some interesting bounds for its residual survival function. Our results generalize several previous findings in this connection. Some illustrative examples are provided.

Keywords: Ageing, Harris family distribution, Marshall-olkin distribution, Reliability function.

1 Introduction

Distributions such as exponential, Weibull, gamma, etc, have a limited range of behavior and can not model all situations found in analyze of lifetime or survival data. This motivated [17] and [8] to develop two new families of distributions. In their approaches, they considered a baseline distribution and extended it to a new and more flexible distribution. The resulting classes are

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called Marshall-Olkin and Harris family of distributions, respectively. Both classes of such distributions are, in particular, useful in reliability theory. To cover a wide range of data such as those with high degrees of skewness and kurtosis, they added a parameter to the model called the tilt parameter.
Recently, another method of constructing Harris family was introduced by [11]. They revealed that Harris family is a proportional failure rate model which is obtained from a modified Marshall-Olkin family.
Due to various reasons, in many practical situations the, tilt parameter may not be constant and the occurrence of heterogeneity is sometimes unpredictable and cannot be explained. For instance, survival analysis is mainly concerned with investigating the hazard of death at any time when an individual patient is involved in a clinical trial or other medical study. Due to the difference between individuals in their susceptibility to causes of death or disease, response to treatment, and influence of various risk factors, the observed covariate, such as demographic, physiological, or lifestyle characteristics, are taken into account. Nevertheless, heterogeneity unexplained by observed covariate usually plays an important role because it sometimes leads to a misleading conclusion (cf. [16]). Therefore, it is important to inspect the unobserved random factors' influence on the random variable (rv). Considering this fact, we need to study the mixture of a family of distributions. Mixture distributions are often used in mixture models, which are used to express probabilities of sub-populations within a larger population. A mixture model can accommodate the historically observed data in that sense and offers a flexible solution for different distributional forms. Recently, [6], [4] and [5] were concerned with stochastic comparison of certain distributions with their mixtures. More recently, [1] [2] were concerned with stochastic comparison of two Harris family distributions having the same and different tilt parameters, respectively.
In this paper, we take the tilt parameter in a Harris family distribution as a rv and construct a mixed Harris family distribution. Mixtures are of interest, e.g., in survival analysis, reliability theory and actuarial sciences.

In a Harris family distribution, there is no theoretical basis for choosing the baseline distribution and the distribution of its tilt parameter, when tilt parameter is a rv. Therefore, it is important to see how a Harris family rv responds to the change of its baseline distribution and tilt parameter. The knowledge of ageing properties of baseline distributions helps to better understand the mechanism of a mixed model. [14] have studied the structure and properties of the proportional odds model. They have shown positive ageing properties are transformed from one rv to another. Recently [11] discussed several results in connection with behavior of the failure rate function for the Harris family and studied certain related stochastic orderings [7]. presented a proportional hazard version of Marshall-Olkin family of distributions and investigated likelihood ratio ordering in this model. [18] discussed closure properties of mixture of the family of distributions under different stochastic orderings. [19] studied some ageing properties of Marshall-Olkin family of distributions. [2] compared Harris family distribution with its mixture with respect to several stochastic orderings. Now, in this paper, we obtain an upper bound for the maximum error in evaluating the reliability function for this model when the baseline distribution function (df) is mistakenly assumed to have a constant hazard rate. In many practical problems, using a sample data set, we are able to obtain some life information such as the mean and variance of a life distribution. But the exact value of the reliability function can not be easily obtained. However, it is still helpful to derive some bounds for a reliability function based on the known information. These bounds can tell us the scope of the reliability of products and provide a basis for further improvements. In addition, we obtain two bounds for survival functions conditioned on the tilt random parameter, which are useful in distinguishing the failure probability of a component after a time t when the tilt parameter is unobservable. We also obtain a bound for the mean of a mixed Harris family in terms of the mean of the baseline distribution. Further, upper and lower bounds are presented for the residual survival function of a mixture of Harris family. Our results enfold all related findings on Marshall-Olkin family. In Section 2, we shall state methods of generating Marshall-Olkin and Harris family distributions. In Section 3, we obtain several interesting bounds for a mixed Harris family which are useful in reliability. Some illustrative examples are provided.

2 Harris family of distributions

Marshall-Olkin and Harris family distributions are basically generated by the same methods. They proceed as follows: Let $X_1, X_2, \ldots$ be a sequence of independently identically distributed (iid) random variables (rv's) with common df F and survival function (sf) $\bar{F} = 1 - F$. Let $X = \min\{X_1, X_2, \ldots, X_N\}$, where N is a positive integer valued rv independent of Y_i's with probability generating function (pgf)

$$P_N(t) = E(t^N) = \sum_{n=0}^{\infty} t^n P(N = n), \quad t \in [0, 1].$$

It is noted that X can be viewed as the lifetime of a series system with iid component lifetimes $X_1, X_2, \ldots, X_N$ and a random number N of components. Clearly, the sf $\bar{H}$ of X has the representation

$$\bar{H}(x) = \sum_{n=0}^{\infty} [\bar{F}(x)]^n P(N = n), \tag{1}$$

so that

$$\bar{H}(x) = P_N(\bar{F}(x)). \tag{2}$$

Assuming N is a geometric rv, [17] introduced the so-called Marshall-Olkin distribution. In particular, by taking $\bar{F}$ as the sf of exponential and Weibull distributions, they introduced Marshall-Olkin extended exponential (MOEE) and Marshall-Olkin extended Weibull (MOEW) models, respectively. The following Harris pgf, introduced by [13], was used by [8]

$$P_N(s; \theta, k) = \left\{ \frac{\theta s^k}{1 - \bar{\theta} s^k} \right\}^{1/k}, \qquad k > 0, \qquad 0 < \theta < 1, \qquad \bar{\theta} = 1 - \theta, \tag{3}$$

to generate the Harris family of survival functions $\bar{H}$ as

$$\bar{H}(x;\theta,k)=\Big(\frac{\theta\bar{F}^k(x)}{1-\bar{\theta}\bar{F}^k(x)}\Big)^{1/k},\qquad k>0,\qquad 0<\theta<\infty\qquad \bar{\theta}=1-\theta. \quad (4)$$

The df F in Equation (Eq) (2) is called the baseline df and θ is called the tilt parameter. It is easy to see that hazard rates corresponding to F and $H(\cdot;\theta,k)$, namely, $r_F=f/\bar{F}$ and $r_H(.;\theta,k)=h(.;\theta,k)/\bar{H}(.;\theta,k)$, are related by

$$r_H(x;\theta,k)=\frac{r_F(x)}{1-\bar{\theta}\bar{F}^k(x)},\qquad -\infty<x<\infty,\qquad 0<\theta<\infty\qquad k>0. \quad (5)$$

Clearly, $r_H(x;\theta,k)$ is greater than $r_F(x)$ when $0<\theta\leq 1$. It is smaller than $r_F(x)$ when $\theta\geq 1$. They are the same when $\theta=1$. We also observe that for $k=1$, pgf (1) reduces to the geometric pgf which leads to Marshall-Olkin distribution.

Recently, another method of constructing Harris family was proposed by [11]. They showed that the Harris family of distributions can also be constructed as a proportional failure rate model. They initially generated a new class of distributions with sf $\bar{F}_1(x)=\bar{F}_0{}^k(x)$, which is a proportional failure rate model of the initial one. Next, they applied Marshall-Olkin's transformation to this new distribution and obtain the following sf:

$$\bar{G}(x)=\frac{\theta\bar{F}_1(x)}{1-\bar{\theta}\bar{F}_1(x)};$$

that is, $\bar{G}(x)$ is the Marshall-Olkin sf having baseline sf $\bar{F}_1(x)$. Finally, they considered the model with sf $\bar{F}(x)=\bar{G}^{\frac{1}{k}}(x)$, which gives the Harris sf.

3 Main results

In Eq (2), let the parameter Θ be an absolutely continuous rv with df $G(.)$ and pdf $g(.)$. Then, its corresponding unconditional Harris sf is given by

$$\bar{H}(x;k)\ =\ \int_0^\infty \bar{H}(x;\theta,k)g(\theta)d\theta,$$

$$= E[\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)}]^{\frac{1}{k}}\bar{F}(x). \quad (6)$$

We denote the corresponding rv by X^*. Clearly pdf of $X^*$is given by

$$\begin{aligned} h(x;k) &= \int_0^\infty h(x;\theta,k)g(\theta)d\theta, \\ &= E[\frac{\Theta}{(1-\bar{\Theta}\bar{F}^k(x))^{k+1}}]^{\frac{1}{k}}f(x). \quad (7) \end{aligned}$$

[19] have investigated some ageing properties of Marshall-Olkin distribution. In what follows, we shall compare, more generally, Harris distributions with their mixtures, i.e.,when $k>0$ is arbitrary . Our results enfold [19]'s findings in this connection.

We now obtain some useful bounds concerning a tilt-mixture Harris family distribution. First, we note that for any x, $t>0$ and $k\geq 1$, $(\frac{t}{1-(1-t)\bar{F}^k(x)})^{\frac{1}{k}}$ is a concave function of t. Thus, using Jensen's inequality, where $\mu = E(\Theta)$, we have

$$\bar{H}(x;k) = \bar{F}(x)E(\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)})^{\frac{1}{k}} \leq \bar{F}(x)(\frac{\mu}{1-(1-\mu)\bar{F}^k(x)})^{\frac{1}{k}}.$$

Lemma 3.1. *For a non-negative baseline rv X, we have*

$$\begin{cases} E(\Theta)^{\frac{1}{k}} \leq \frac{\bar{H}(x;k)}{\bar{F}(x)} \leq 1, & P(0<\Theta<1)=1 \\ 1 \leq \frac{\bar{H}(x;k)}{\bar{F}(x)} \leq E(\Theta)^{\frac{1}{k}}, & P(\Theta\geq 1)=1 \end{cases}$$

The next theorem gives an upper bound for the maximum error in evaluating the reliability function, when the baseline df is mistakenly assumed to have a constant hazard rate. Before this, we need to recall the concept of IMRL (DMRL). If the life of a component is represented by a rv X, then X is said to be IMRL (DMRL) if its mean residual life function $m(t) = E(X-t|X>t)$ is increasing (decreasing) in $t\geq 0$. In life testing situations, the concept of MRL is employed. For example, when new components are initially produced, many may fail before a time $t>0$. In such situations, IMRL (DMRL) is of interest.

Theorem 3.2. *Let the baseline rv X be non-negative and* $\bar{H}_0(x;k) = e^{-\frac{x}{\mu_0}}E(\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)})^{\frac{1}{k}}$, *where* $\mu_0 = E(X)$. *If the baseline distribution has DMRL property, then*

$$sup_{x\geq 0} \mid \bar{H}(x;k) - \bar{H}_0(x;k) \mid \leq \begin{cases} E(\Theta)^{\frac{1}{k}}(1-e^{-1})(1-\gamma_0^2), & P(\Theta \geq 1) = 1 \\ (1-e^{-1})(1-\gamma_0^2), & P(0 < \Theta < 1) = 1 \end{cases}$$

where γ_0 *is the coefficient of variation of the baseline distribution.*

In the following theorem we obtain an upper bound for the failure probability of a component after a time t. But, first we need the definition of new better than used in the second stochastic dominance distributions (NBU(2)) property. NBU(2) is the class of NBU in the increasing concave (icv) order (see [21]). This class emphasizes that a new item has a larger life length in the (icv) ordering than does a used one at age $t > 0$. For more details on NBU(2), see [15]. More precisely, we have

Definition 3.3. Let $X_t = (X - t \mid X > t)$ be the residual life of X at time $t \geq 0$. X is said to be NBU(2) if $X \geq_{icv} X_t$ or, equivalently, if $\int_0^x \bar{F}_{X_t}(u)du \leq \int_0^x \bar{F}_X(u)du$ for all $x \geq 0$.

Theorem 3.4. *Let tilt parameter* Θ *have NBU(2) property and* $k \geq 1$. *For non-negative rv* X*,*
i) Provided that $P(0 \leq \Theta \leq 1) = 1$*, we have*

$$P(X^* \geq t \mid \Theta > \vartheta) \leq min\{\bar{F}(t) + (\frac{\bar{F}^k(t)\vartheta}{1-(1-\vartheta)\bar{F}^k(t)})^{\frac{1}{k}},\ 1\}.$$

ii) Provided that $P(\Theta \geq 1) = 1$*, we have*

$$P(X^* \geq t \mid \Theta > \vartheta) \leq min\{\bar{F}(t)E(\Theta^{\frac{1}{k}}) + (\frac{\bar{F}^k(t)\vartheta}{1-(1-\vartheta)\bar{F}^k(t)})^{\frac{1}{k}},\ 1\}.$$

Remark 3.5. Clearly, taking $k = 1$, Theorem 3.4 reduces to Theorem 4.1 of [16].

Example 3.6. Let X be an exponential rv with sf $\bar{F}(x) = exp(-x)$ for $x > 0$ and let Θ be a rv with sf $\bar{G}(\theta) = \frac{e^{-2\theta+2}-1}{e^2-1}$ for $0 < \theta < 1$. Clearly, Θ is NBU(2). For $k = 2$, Figure 1 shows that

$$P(X^* \geq t \mid \Theta > v) = \frac{2e^{-t}}{e^{-2v}-e^{-2}} \int_v^1 \frac{u^{0.5}e^{-2u}}{(1-(1-u)e^{-2t})^{0.5}}du$$

$$\leq \; min\{\bar{F}(t)+(\frac{\bar{F}^k(t)\vartheta}{1-(1-\vartheta)\bar{F}^k(t)})^{\frac{1}{k}},\ 1\}$$
$$= \; min\{e^{-t}+(\frac{e^{-2t}\nu}{1-(1-\nu)e^{-2t}})^{0.5},\ 1\}.$$

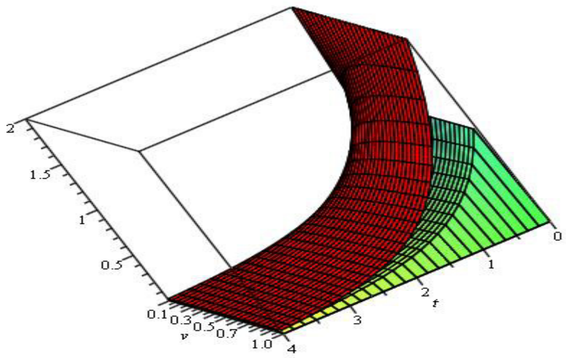

Figure 1: The green color is the plot of $P(X^* \geq t \mid \Theta > \nu)$ and the red color is the plot of $min\{\bar{F}(t)+(\frac{\bar{F}^k(t)\vartheta}{1-(1-\vartheta)\bar{F}^k(t)})^{\frac{1}{k}},\ 1\}$

The following theorem gives certain bounds for sf of the residual rv X_t^* in terms of that of X.

Theorem 3.7. *For non-negative rv X,*

$$min\{E(\Theta^{1/k}|X^* > t), 1\}\bar{F}_t(x) \leq \bar{H}_t(x;k) \leq max\{E(\Theta^{1/k}|X^* > t), 1\}\bar{F}_t(x).$$

Remark 3.8. Clearly, taking $k = 1$, Theorem 3.7 reduces to the results of Theorem 2.2 of [19] and taking Θ as a degenerate rv and $k = 1$, it reduces to the results of Proposition 9 of [14].

In the following theorem we obtain a bound for the expectation of a Harris family mixture. But, first we need the following lemma and the definition of harmonically new better than used in expectation (HNBUE) (harmonically new worse than used in expectation (HNWUE)) property. HNBUE (HNWUE) is a quantity for describing the aging property of a system. For more details on HNBUE (HNWUE), see [10]. More precisely, we have

Definition 3.9. Let $E(X) = \mu$. X is said to belong to the class of HNBUE (HNWUE) if and only if, for all $x > 0$,

$$\int_x^\infty \bar{F}(u)du \leq (\geq)\mu\ exp(\frac{-x}{\mu}).$$

Lemma 3.10. *The pdf corresponding to Harris pgf in Eq (1) is*

$$P(N=n)=\theta^{\frac{1}{k}}\bar{\theta}^{n}\frac{\Gamma(n+\frac{1}{k})}{n!\,\Gamma(\frac{1}{k})} \quad n=0,1,..., \quad 0<\theta<1, \quad k>0 \quad integer, \quad (8)$$

The above pdf together with its properties were also studied by [20].

Theorem 3.11. *Let $P(0<\Theta\leq 1)=1$ and $k>0$ be an integer.*
If X is HNBUE, then,

$$E(X^*)\geq \frac{E(X)}{k\Gamma(\frac{1}{k})}E[\Theta^{\frac{1}{k}}(\frac{\Theta ln(\Theta)}{\bar{\Theta}^2}+\frac{1}{\bar{\Theta}}-\frac{1}{2})].$$

Example 3.12. Let X be a rv having sf $\bar{F}(x)=exp(-2x)$ for $x>0$. Clearly, X is HNBUE. Let Θ be a rv with pdf $g(\theta)=3(1-\theta)^2$ on $(0,1)$. Then, using Maple software, for $k=2$, we obtain

$$\begin{aligned} E[\Theta^{0.5}(\frac{\Theta ln(\Theta)}{\bar{\Theta}^2}+\frac{1}{\bar{\Theta}}-\frac{1}{2})] &= \int_0^1 3\theta^{0.5}\bar{\theta}^2(\frac{\theta ln(\theta)}{\bar{\theta}^2}+\frac{1}{\bar{\theta}}-\frac{1}{2})d\theta \\ &\simeq 0.091 \end{aligned}$$

Hence,

$$\frac{E(x)}{2\Gamma(\frac{1}{2})}E[\Theta^{0.5}(\frac{\Theta ln(\Theta)}{\bar{\Theta}^2}+\frac{1}{\bar{\Theta}}-\frac{1}{2})]\simeq 0.012$$

and

$$\begin{aligned} E(X^*) &= \int_0^\infty \bar{H}(x;2)dx \\ &= \int_0^\infty\int_0^1 \frac{3exp(-2x)\theta^{\frac{1}{2}}(1-\theta)^2}{(1-(1-\theta)exp(-4x))^{\frac{1}{2}}}d\theta dx \\ &\simeq 0.272 \end{aligned}$$

which confirms that $E(X^*)\geq \frac{E(X)}{k\Gamma(\frac{1}{k})}E[\Theta^{\frac{1}{k}}(\frac{\Theta ln(\Theta)}{\bar{\Theta}^2}+\frac{1}{\bar{\Theta}}-\frac{1}{2})]$.

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Inference for a Simple Step-stress Model with Progressively Type-II Censored Competing Risks Data Under the Exponential Distribution

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Abstract: In this article, a simple step-stress life-testing experiment in the presence of exponentially distributed competing risks is considered. After observing a pre-determined number of failures, the stress of the experiment is changed. Also, in order to save cost and time, a progressive Type-II censoring scheme is conducted. Under this setup, the maximum likelihood estimators of the unknown parameters and the exact conditional distributions of the maximum likelihood estimators are discussed.

Keywords: Competing risks model, Exponential distribution, Progressive Type-II censoring, Step-stress accelerated life-testing.

1 Introduction

In many situations in reliability and survival analysis, because of high reliability of some products, the lifetime of products under normal operating conditions may not terminate at an adequate time. In such situations, the standard life-testing methods are not appropriate and accelerated life-testing (ALT) experiments can be used in order to cause rapid failures. Such experiments consist in forcing the products to fail more quickly than under normal operating conditions. For more details on different ALT models, the reader can be

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referred to Nelson and Meeker [20], Nelson [22], Meeker and Escobar [19] and Bagdonavicius and Nikulin [1]. We focus here on a special case of ALT known as step-stress testing for which the experimenter chooses one or more stress factors in life-testing experiment and change the stress levels during the experiment at pre-specified times or upon the occurrence of a fixed number of failures. Examples of stress factors include load, pressure, temperature, voltage or a combination of these factors that directly reduces the lifetime of products to failure. We consider here a simple step-stress model involving only two stress levels and the stress changes when a pre-specified number of failures takes place. It is further assumed that the time-to-failure data come from a cumulative damage or cumulative exposure model. It is one of the fundamental models in step-stress testing as introduced by Sedyakin [25]. This model that connects between the lifetime distribution of units at successive stress levels, has been further examined and generalized by Bagdonavicius [2] and Nelson [21]. From the assumptions of cumulative exposure model and exponentially distributed life, the literature on the simple step-stress model is rather extensive and the reader is referred to Balakrishnan et al. [7], Balakrishnan and Han [5], Balakrishnan and Xie [8, 9], Balakrishnan et al. [10] in the presence of different censoring schemes.

In reliability theory and survival analysis, when there exist more than one cause of failure (defined as risk factor), assessing the lifetime of products with an isolated risk factor is not usually possible. Hence, the experimenter needs to assess the effect of each risk factor in the presence of other risk factors. In such a situation, the experimenter encounters the problem of competing risks. For example, the failure of a bearing assembly may be related to bearing failure or shaft failure. The data for such a competing risks model must come in a bivariate form composed of the lifetime of the unit and an indicator variable denoting which risk factor occurred for the unit. In practice, the risk factors may be statistically independent or dependent. In most situations, however, for analyzing a competing risks model, the risk factors are assumed to be independent. See, for example, David and Moeschberger [12], Crowder [11] and Sarhan et al. [24]. Here, we consider a problem of competing risks involving two risk factors which are statistically independent. Many researchers dealt with the statistical inferences based on the simple step-stress model in the presence of competing risks. For example, assuming the exponential lifetime distribution for competing risks, we refer the readers to Balakrishnan and Han [4], Han and Balakrishnan [15] and Ganguly and Kundu [14]. Also, Liu and Shi [18] focused on the inference for a simple step-stress model with progressively censored competing risks data from Weibull distribution.

In life testing expriments, when for saving time or cost reduction, the complete lifetime of all products on test can not be observed, therefore censored samples may occur. The most common censoring scheme is Type-II censoring in which the experiment is terminated after observing a pre-specified number of failed units. Progressive Type-II censoring that provides higher flexibility to the experimenter in the design stage, allows to continual removal of a pre-specified number of un-failed units on test at non-terminal time points. This allowance may be desirable when a compromise between the reduced time of the experiment and the observation of at least some extreme lifetimes is sought. As a prominent work on progressive censoring, we refer the readers to Balakrishnan and Aggarwala [3]. This censoring scheme can be described as follows. Consider n identical units being put on a life test at time zero. The life test terminates as soon as the rth failure is observed and all remaining units are removed from the life test. When the ith unit fails $(i = 1,\dots,r)$, R_i randomly selected units from surviving ones are removed from the life test. Note that the values of r and $(R_1,\dots,R_r)$ are pre-specified and that $\sum_{i=1}^{r} R_i = n-r$. As a special case, if $R_1 = \cdots = R_{r-1} = 0$ and $R_r = n-r$, the progressive censoring scheme is reduced to the Type-II censoring scheme, while after running a progressive censoring scheme with $R_1 = \cdots = R_m = 0$, the un-censored lifetime data are observed. The progressively censored data have been used by many researchers for parametric inferences on various lifetime distributions.

For example, see Balakrishnan et al. [6], Helu and Samawi [16], Fernández [13], Khorram and Farahani [17] and Panahi and Asadi [23].

In this paper, we consider the simple step-stress model under a progressive Type-II censoring scheme when the lifetime distributions of different two risk factors are independently exponentially distributed. In the Section 3, we present a brief overview of the considered model and obtain the maximum likelihood estimators (MLEs) of the unknown parameters. We show that these MLEs do not always exist. Therefore, the exact conditional distributions of these MLEs are discussed in Section 4.

2 Model description and MLE

A random sample of n identical units is put on a simple step-stress ALT experiment at the initial stress level s_1. The successive failure times are then recorded along with the information about which risk factor caused each failure. When r units fail, the experiment termites. At the time of r_1th failure, the stress level is increased to s_2 and the life test continues until the r_2 failures under the stress level s_2 are observed. Note that r_1 and r_2 are pre-fixed and that $r_1 + r_2 = r$, $r_1 \geq 1$, $r_2 \geq 1$ and $2 \leq r \leq n$. According to the progressive Type-II censoring scheme, at the time of ith failure ($i = 1, \ldots, r$), R_i units are removed from the experiment. Let the recorded data be denoted by $(x_1, \delta_1), \ldots, (x_r, \delta_r)$ where $x = (x_1, \cdots, x_r)$ are the observed failure times and $\delta = (\delta_1, \ldots, \delta_r)$ are the observed sequence of the cause of failures with $\delta_i = 0$ or 1 if ith failure occurs due to first or second risk factor, respectively. Also, assume that the time-to-failure by each competing risk follows an independent exponential distribution which obeys the cumulative exposure model. Let θ_{ij} be the mean time-to-failure of a unit at the stress level s_i by the risk factor j, with $i, j = 1, 2$. Then, if τ is the changing time of the stress level from s_1 to s_2, the cumulative distribution function (CDF) and probability density function (PDF) of the lifetime of units due to the first risk factor are given, respectively, by

$$F(x)=\begin{cases}F_1(x)=1-\exp\{-\frac{x}{\theta_{11}}\}, & \text{if } 0<x<\tau,\\ F_2(x)=1-\exp\{-\frac{\tau}{\theta_{11}}-\frac{x-\tau}{\theta_{21}}\} & \text{if } \tau\le x<\infty,\end{cases} \tag{1}$$

and

$$f(x)=\begin{cases}f_1(x)=\frac{1}{\theta_{11}}\exp\{-\frac{x}{\theta_{11}}\}, & \text{if } 0<x<\tau,\\ f_2(x)=\frac{1}{\theta_{21}}\exp\{-\frac{\tau}{\theta_{11}}-\frac{x-\tau}{\theta_{21}}\} & \text{if } \tau\le x<\infty.\end{cases} \tag{2}$$

Similarly, the CDF and PDF of the lifetime of units due to the second risk factor are given, respectively, by

$$G(x)=\begin{cases}G_1(x)=1-\exp\{-\frac{x}{\theta_{12}}\}, & \text{if } 0<x<\tau,\\ G_2(x)=1-\exp\{-\frac{\tau}{\theta_{12}}-\frac{x-\tau}{\theta_{22}}\} & \text{if } \tau\le x<\infty,\end{cases} \tag{3}$$

and

$$g(x)=\begin{cases}g_1(x)=\frac{1}{\theta_{12}}\exp\{-\frac{x}{\theta_{12}}\}, & \text{if } 0<x<\tau,\\ g_2(x)=\frac{1}{\theta_{22}}\exp\{-\frac{\tau}{\theta_{12}}-\frac{x-\tau}{\theta_{22}}\} & \text{if } \tau\le x<\infty.\end{cases} \tag{4}$$

Letting $\theta=(\theta_1,\theta_2)$ with $\theta_i=(\theta_{i1},\theta_{i2})$ for $i=1,2$, it can be shown that the likelihood function of θ based on the observed data (x,δ) is formulated as

$$\begin{aligned}L(\theta;x,\delta)=C\prod_{i=1}^{r_1}\left[f_1(x_i)\bar{G}_1(x_i)\right]^{\delta_i}\left[g_1(x_i)\bar{F}_1(x_i)\right]^{1-\delta_i}\left[\bar{F}_1(x_i)\bar{G}_1(x_i)\right]^{R_i}\\ \times\prod_{i=r_1+1}^{r}\left[f_2(x_i)\bar{G}_2(x_i)\right]^{\delta_i}\left[g_2(x_i)\bar{F}_2(x_i)\right]^{1-\delta_i}\left[\bar{F}_2(x_i)\bar{G}_2(x_i)\right]^{R_i}.\end{aligned} \tag{5}$$

for $0<x_1<\cdots<x_r<\infty$, where

$$C=n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-r+1-\sum_{k=1}^{r-1}R_k).$$

Substituting the Equations (1)-(4) in Equation (5), the likelihood function of θ under the exponential distribution is derived as

$$\begin{aligned}L(\theta;x,\delta)=&C\lambda_{11}^{m_1}\lambda_{12}^{r_1-m_1}\lambda_{21}^{m_2}\lambda_{22}^{r_2-m_2}\\ &\times\exp\{-(\lambda_{11}+\lambda_{12})U_1\}\exp\{-(\lambda_{21}+\lambda_{22})U_2\},\end{aligned} \tag{6}$$

where $m_1 = \sum_{i=1}^{r_1} \delta_i$, $m_2 = \sum_{i=r_1+1}^{r} \delta_i$, $\lambda_{ij} = 1/\theta_{ij}$, for $i, j = 1, 2$,

$$D_1 = \sum_{i=1}^{r_1} x_i(R_i + 1) + \Big(n - \sum_{i=1}^{r_1}(R_i + 1)\Big)x_{r_1},$$

and

$$D_2 = \sum_{i=r_1+1}^{r} (x_i - x_{r_1})(R_i + 1).$$

Note that m_1 and m_2 are the number of units that fail due to the first risk factor at the stress levels s_1 and s_2, respectively and hence $r_1 - m_1$ and $r_2 - m_2$ are the number of units that fail due to the second risk factor at the stress levels s_1 and s_2, respectively. We consider $M = (M_1, M_2)$ as the random vector corresponding to the observed integer vector $m = (m_1, m_2)$. Also, D_1 and D_2 are the *Total Time on Test* statistic at the stress levels s_1 and s_2, respectively.

From the likelihood function in Equation (6), it is clearly observed that the MLEs of all unknown parameters θ exist if $0 < M_1 < r_1$ and $0 < M_2 < r_2$. That is, at least one failure caused by each risk factor must be occurred at each stress level for estimating the parameters θ simultaneously. In this situation, the log-likelihood function of θ is obtained from Equation (6) as

$$l(\theta; x, \delta) = \log C + m_1 \log \lambda_{11} + (r_1 - m_1) \log \lambda_{12} + m_2 \log \lambda_{21} + (r_2 - m_2) \log \lambda_{22} \\ - (\lambda_{11} + \lambda_{12}) D_1 - (\lambda_{21} + \lambda_{22}) D_2,$$

from which the MLEs of θ_{11}, θ_{12}, θ_{21}, and θ_{22} are readily obtained as

$$\hat{\theta}_{11} = \frac{D_1}{M_1}, \quad \hat{\theta}_{12} = \frac{D_1}{r_1 - M_1}, \quad \hat{\theta}_{21} = \frac{D_2}{M_2}, \quad \text{and} \quad \hat{\theta}_{22} = \frac{D_2}{r_2 - M_2}, \tag{7}$$

respectively.

3 Conditional Distributions of the MLEs

In order to find the exact conditional distributions of $\hat{\theta}_{ij}$ for $i, j = 1, 2$, we obtain the conditional moment generating function (CMGF) of $\hat{\theta}_{ij}$, conditioned on the event $\{0 < M_1 < r_1, 0 < M_2 < r_2\}$. Frist, we need the following lemma.

Lemma 3.1. *Let* (X,Δ) *with* $X=(X_1,\dots,X_r)$ *and* $\Delta=(\Delta_1,\dots\Delta_r)$ *be the progressively Type-II censored competing risks sample from cumulative exposure PDFs* $f(x)$ *and* $g(x)$ *given in* (2) *and* (4)*, respectively. the joint probability mass function of* M *is given by*

$$P(M=m)=\prod_{i=1}^{2}\binom{r_i}{m_i}p_i^{m_i}(1-p_i)^{r_i-m_i},\ \ m_1=0,\dots,r_1,\ m_2=0,\dots,r_2,$$

where $p_i=\theta_{i2}/(\theta_{i1}+\theta_{i2})$ *for* $i=1,2$*. That is,* M_1 *and* M_2 *are independent binomial random variables with parameters* (r_1,p_1) *and* (r_2,p_2)*, respectively.*

Proof. From Equation (6), we have

$$\begin{aligned}P(M=m)=&\sum_{\delta_1\in Q_1}\sum_{\delta_2\in Q_2}\int_0^\infty\cdots\int_{x_{r-1}}^\infty f_{X,\Delta}(x,\delta)\,\mathrm{d}x_r\cdots\mathrm{d}x_1\\ =&C\binom{r_1}{m_1}\binom{r_2}{m_2}\lambda_{11}^{m_1}\lambda_{12}^{r_1-m_1}\lambda_{21}^{m_2}\lambda_{22}^{r_2-m_2}\\ &\times\int_0^\infty\cdots\int_{x_{r-1}}^\infty\prod_{i=1}^{2}\exp\{-(\lambda_{i1}+\lambda_{i2})U_i\}\,\mathrm{d}x_r\cdots\mathrm{d}x_1\\ =&\prod_{i=1}^{2}\binom{r_i}{m_i}p_i^{m_i}(1-p_i)^{r_i-m_i},\end{aligned}$$

where

$$Q_1=\left\{\delta_1=(\delta_1,\dots,\delta_{r_1}):\ \delta_i=0,1\text{ for }i=1,\dots,r_1,\text{ and }\sum_{i=1}^{r_1}\delta_i=m_1\right\},$$

and

$$Q_2=\left\{\delta_2=(\delta_{r_1+1},\dots,\delta_r):\ \delta_i=0,1\text{ for }i=r_1+1,\dots,r,\text{ and }\sum_{i=r_1+1}^{r}\delta_i=m_2\right\}.$$

□

Theorem 3.2. *The conditional PDFs of* $\hat{\theta}_{11}$ *and* $\hat{\theta}_{12}$*, conditioned on* $0<M_1<r_1$*, are, respectively, given by*

$$f_{\hat{\theta}_{11}}(x)=\frac{1}{1-p_1^{r_1}-(1-p_1)^{r_1}}\sum_{m_1=1}^{r_1-1}\binom{r_1}{m_1}p_1^{m_1}(1-p_1)^{r_1-m_1}\gamma\Big(x;r_1,m_1(\lambda_{11}+\lambda_{12})\Big),\quad(8)$$

and

$$f_{\hat{\theta}_{12}}(x) = \frac{1}{1-p_1^{r_1}-(1-p_1)^{r_1}} \sum_{m_1=1}^{r_1-1} \binom{r_1}{m_1} p_1^{m_1}(1-p_1)^{r_1-m_1} \gamma\Big(x; r_1, (r_1-m_1)(\lambda_{11}+\lambda_{12})\Big), \tag{9}$$

where

$$\gamma(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x};\; x>0,$$

is the gamma PDF with parameters $\alpha>0$ *and* $\beta>0$.

Proof. Similar to the Equation (6), the joint PDF of (X_1,Δ_1) with $X_1=(X_1,\ldots,X_{r_1})$ and $\Delta_1=(\Delta_1,\ldots,\Delta_{r_1})$ is readily obtained to be

$$f_{X_1,\Delta_1}(x_1,\delta_1) = C_1 \lambda_{11}^{m_1} \lambda_{12}^{r_1-m_1} \exp\{-(\lambda_{11}+\lambda_{12})U_1\}, \tag{10}$$

where $0<x_1<\cdots<x_{r_1}$, $\delta_i\in\{0,1\}$ for $i=1,\ldots,r_1$ and that

$$C_1 = n(n-1-R_1)(n-2-R_1-R_2)\cdots(n-r_1+1-\sum_{k=1}^{r_1-1} R_k).$$

Denoting $\mathrm{M}_{\hat{\theta}_{11}}(t)$ for the CMGF of $\hat{\theta}_{11}$, conditioned on $\{0<M_1<r_1\}$, we can write

$$\begin{aligned}
\mathrm{M}_{\hat{\theta}_{11}}(t) &= \mathrm{E}(e^{t\hat{\theta}_{11}}|0<M_1<r_1)\\
&= \sum_{m_1=1}^{r_1-1} \mathrm{E}(e^{t\hat{\theta}_{11}}|M_1=m_1)\Pr(M_1=m_1|0<M_1<r_1)\\
&= \sum_{m_1=1}^{r_1-1}\sum_{\delta_1\in Q_1} \mathrm{E}(e^{t\hat{\theta}_{11}}|\Delta_1=\delta_1)\Pr(\Delta_1=\delta_1|M_1=m_1)\Pr(M_1=m_1|0<M_1<r_1)\\
&= \frac{1}{\Pr(0<M_1<r_1)} \sum_{m_1=1}^{r_1-1}\sum_{\delta_1\in Q_1} \int_0^\infty \cdots \int_{x_{r_1-1}}^\infty e^{tD_1/m_1} f_{X_1,\Delta_1}(x_1,\delta_1)\, \mathrm{d}x_{r_1}\cdots \mathrm{d}x_1.
\end{aligned}$$

Upon substituting $f_{X_1,\Delta_1}(x_1,\delta_1)$ from Equation (1) into the above expression and carrying out the necessary integrations, we obtain

$$\mathrm{M}_{\hat{\theta}_{11}}(t) = \frac{1}{1-p_1^{r_1}-(1-p_1)^{r_1}} \sum_{m_1=1}^{r_1-1} \binom{r_1}{m_1} p_1^{m_1}(1-p_1)^{r_1-m_1} \left(1-\frac{t}{m_1(\lambda_{11}+\lambda_{12})}\right)^{-r_1},$$

for $t<\lambda_{11}+\lambda_{12}$. Applying the inversion theorem of a moment generating function, the conditional PDF of $\hat{\theta}_{11}$ in Equation (1) is derived. Similarly, the conditional PDF of $\hat{\theta}_{12}$ in Equation (2) is proved which the respective proof is omitted for brevity. □

Theorem 3.3. *The conditional PDFs of* $\hat{\theta}_{21}$ *and* $\hat{\theta}_{22}$*, conditioned on* $0 < M_2 < r_2$*, are, respectively, given by*

$$f_{\hat{\theta}_{21}}(x) = \frac{1}{1-p_2^{r_2}-(1-p_2)^{r_2}} \sum_{m_2=1}^{r_2-1} \binom{r_2}{m_2} p_2^{m_2}(1-p_2)^{r_2-m_2}\gamma\Big(x; r_2, m_2(\lambda_{21}+\lambda_{22})\Big), \tag{11}$$

and

$$f_{\hat{\theta}_{22}}(x) = \frac{1}{1-p_2^{r_2}-(1-p_2)^{r_2}} \sum_{m_2=1}^{r_2-1} \binom{r_2}{m_2} p_2^{m_2}(1-p_2)^{r_2-m_2}\gamma\Big(x; r_2, (r_2-m_2)(\lambda_{21}+\lambda_{22})\Big). \tag{12}$$

Proof. From Equations (6) and (1), the conditional joint PDF of (X_2, Δ_2), conditioned on (X_1, Δ_1), with $X_2 = (X_{r_1+1}, \ldots, X_r)$ and $\Delta_2 = (\Delta_{r_1+1}, \ldots, \Delta_r)$ is readily obtained to be

$$f_{X_2,\Delta_2}(x_2, \delta_2) = C_2 \lambda_{21}^{m_2} \lambda_{22}^{r_2-m_2} \exp\{-(\lambda_{21}+\lambda_{22})U_2\}, \tag{13}$$

where $0 < x_{r_1+1} < \cdots < x_r$, $\delta_i \in \{0,1\}$ for $i = r_1+1, \ldots, r$ and that

$$C_2 = (n - r_1 - \sum_{k=1}^{r_1} R_k)(n - r_1 - 1 - \sum_{k=1}^{r_1+1} R_k) \cdots (n - r + 1 - \sum_{k=1}^{r-1} R_k).$$

Denoting $\mathrm{M}_{\hat{\theta}_{21}}(t)$ for the CMGF of $\hat{\theta}_{21}$, conditioned on $\{0 < M_2 < r_2\}$, we can write

$$\begin{aligned}
\mathrm{M}_{\hat{\theta}_{21}}(t) &= \mathrm{E}(e^{t\hat{\theta}_{21}} | 0 < M_2 < r_2) \\
&= \sum_{m_2=1}^{r_2-1} \mathrm{E}(e^{t\hat{\theta}_{21}} | M_2 = m_2) \mathrm{Pr}(M_2 = m_2 | 0 < M_2 < r_2) \\
&= \sum_{m_2=1}^{r_2-1} \sum_{\delta_2 \in Q_2} \mathrm{E}(e^{t\hat{\theta}_{21}} | \Delta_2 = \delta_2) \mathrm{Pr}(\Delta_2 = \delta_2 | M_2 = m_2) \mathrm{Pr}(M_2 = m_2 | 0 < M_2 < r_2) \\
&= \frac{1}{\mathrm{Pr}(0 < M_2 < r_2)} \sum_{m_2=1}^{r_2-1} \sum_{\delta_2 \in Q_2} \int_{x_{r_1}}^{\infty} \cdots \int_{x_{r-1}}^{\infty} e^{tD_2/m_2} f_{X_2,\Delta_2}(x_2, \delta_2)\, \mathrm{d}x_r \cdots \mathrm{d}x_{r_1+1}.
\end{aligned}$$

Upon substituting $f_{X_2,\Delta_2}(x_2,\delta_2)$ from Equation (5) into the above expression and carrying out the necessary integrations, we obtain

$$\mathrm{M}_{\hat{\theta}_{21}}(t) = \frac{1}{1-p_2^{r_2}-(1-p_2)^{r_2}} \sum_{m_2=1}^{r_2-1} \binom{r_2}{m_2} p_2^{m_2}(1-p_2)^{r_2-m_2} \left(1-\frac{t}{m_2(\lambda_{21}+\lambda_{22})}\right)^{-r_2},$$

for $t < \lambda_{21}+\lambda_{22}$. Applying the inversion theorem of a moment generating function, the conditional PDF of $\hat{\theta}_{21}$ in Equation (2) is derived. Similarly, the conditional PDF of $\hat{\theta}_{22}$ in Equation (3) is proved which the respective proof is omitted for brevity. □

From Theorems 3.2 and 3.3, we conclude that the conditional distribution of $\hat{\theta}_{ij}$ for $i,j=1,2$, given $0<M_i<r_i$, does not depend on the values $(R_1,\ldots,R_r)$ which is based on the lack of memory property of exponential distribution.

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An Approach to Scheduling Maintenance and Inspection Planning of Load Sharing k-out-of-n Systems

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Abstract: This paper presents a copula-based approach to scheduling maintenance and inspection planning for load-sharing k-out-of-n systems. The system is monitored at periodic times and corrective and preventive maintenance actions are carried out in response to the observed state process $X(t)$ describing the total number of failed components (decision variable). Assuming a threshold-type policy, the paper aims at minimizing the long-run average maintenance cost rate by determining appropriate inspection intervals and the preventive maintenance threshold. We illustrate the procedure for the case when the components' lifetimes conform to a Weibull distribution.

Keywords: Generic FGM copula model, Inspection, Load-sharing k-out-of-n system, Maintenance.

1 Introduction

In the last three decades the bulk of studies have been performed on both reliability modelling and maintenance optimisation problem for deteriorating multi-component systems. Existing models either turn their attention to inspection policy or preventive replacement policy, or have neglected failure interaction of components [1, 3, 6], which, nevertheless, is often not the case.

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This paper aims to develop these two themes through the use of a developed FGM copula function. The approach explored here in preference to models [4, 5, 7, 8, 9, 10] allows not only the reliability modeling of a load-sharing k-out-of-n system, but also the joint determination of optimal inspection and preventive replacement policy for such systems. Also, it encompasses and examines in a unifying model some of the characteristics that have not been addressed or previously studied in isolation.

2 Assumption

The model is developed in the following setting:

- The lifetimes of components are positively dependent and modelled through a generic FGM copula function [2].
- The decision maker inspects the system according to policy $\Pi = \{k\tau : k \in \mathbb{N}\}$.
- The number of failed components is adopted as a basis for decision making.
- The decision maker's actions includes two types of actions: (i) no action and (ii) perfect repair.
- The lifetimes of components conform to a Weibull distribution with the shape parameter and the scale parameter α and β.

3 Model

The system maintained is a load-sharing k-out-of-n system. The common feature of such systems is that (i) the failure of components leads to an increased tendency for remaining surviving components to fail and (ii) they fails if at least k of the n components fail. In present setting, the components' lifetimes $\mathbf{T} = (T_1, T_2, \cdots, T_n)$ are modelled by an n-variate FGM copula function

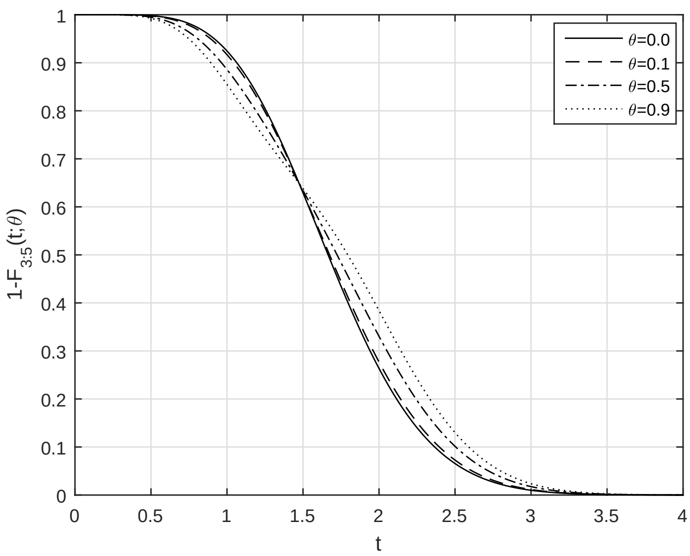

Figure 1: The survival function $\bar{F}_{3:5}(t;\theta)$ for different $\theta \in \{0, 0.1, 0.5, 0.9\}$.

$C(u_1, u_2, \cdots, u_n; \theta)$:

$$\mathrm{C}\left(u_1, u_2, \ldots, u_{\mathrm{n}}; \theta\right) = \prod_{j=1}^{n} u_j \times \left(1 + \theta \sum_{1 \leq j < k \leq n} \left(1 - u_j\right)\left(1 - u_k\right) + \right.$$

$$\left. \theta \sum_{1 \leq j < k < l \leq n} \left(1 - u_j\right)\left(1 - u_k\right)\left(1 - u_l\right) + \ldots + \theta\left(1 - u_1\right)\left(1 - u_2\right)\ldots\left(1 - u_n\right)\right)$$

Proposition 3.1. *Let $\bar{F}_{k:n}(t;\theta)$ denote the survival function of the k-order statistics $T_{k:n}$ of lifetimes $T_1, T_2, \cdots, T_n$ given the dependence factor $\theta \in [0,1]$, then*

$$\bar{F}_{k:n}(t;\theta) = \sum_{r=0}^{k-1} \binom{n}{r} F^r(t) \bar{F}^{(n-r)}(t) \left(1 + \theta \psi_n(r;t)\right) \tag{1}$$

where

$$\psi_n(r;t) = \sum_{v=2}^{n} \sum_{A(v)} \binom{x_1}{n-r} \binom{x_2}{r} (-1)^{x_1} F^{x_1}(t) \bar{F}^{x_2}(t)$$

and $A(v)$ is the set of non-negative integer solutions to the equation $x_1 + x_2 = v$.

Proof. For proof see Ref. [2]. □

Figure 1 illustrates the behavior of the survival function of an 3-out-of-5 system for different dependence degrees $\theta \in \{0, 0.1, 0.5, 0.9\}$.

In this context, the total number of failed components (system state) described by a counting process $X(t)$ $(t \geq 0)$ is adopted as a basis for maintenance decision making. To reveal the true state of components, the decision maker inspects the system according to a periodic policy $\Pi = \{k\tau : k = 1, 2, \cdots\}$. The actions taken after an inspection are completely determined by the system state $X(t)$ observed in one of exclusive subsets $\mathbb{A}_i(\ell)$ $(i = 0, 1)$ and $\mathbb{A}_2$. Let the starting state of the system be $X(0) = x$, that is, the system starts operating with $n - x$ components $(x = 0, 1, \cdots, k-1)$. Then the rules used are that (i) no action (a_0) is taken if the system state is found in subset $\mathbb{A}_0(\ell) = \{x, x+1, \cdots, \ell-1\}$; (ii) it is returned to the 'good as new' state (preventive replacement) (a_1) if the system is observed in subset $\mathbb{A}_1(\ell) = \{\ell, \ell+1, \cdots, k-1\}$ and (iii) it is replaced on failure (a_2) $(X(t) \in \mathbb{A}_2 = \{k, k+1, \cdots, n\})$. As noted, the decision process is thus driven by the excursions of the state process $X(t)$ into the state space Ω partitioned into non-overlapping sets $\mathbb{A}_i$: $\Omega = \mathbb{A}_0(\ell) \cup \mathbb{A}_1(\ell) \cup \mathbb{A}_2$. If the doubleton $\langle a, X(t) \rangle$ denotes the action taken in response to the underlying process $X(t)$, then

$$\langle a, X(t) \rangle = \begin{cases} a_0, & X(t) \in \mathbb{A}_0; \\ a_1, & X(t) \in \mathbb{A}_1; \\ a_2, & X(t) \in \mathbb{A}_2. \end{cases}$$

4 Long-run average cost rate

In present approach, the average cost rate is adopted as a measure of policy for optimizing maintenance policies determined by the optimal period of inspection τ^* and the optimal replacement threshold ℓ^*. The approach rests on the identification of an embedded renewal process defined by failure epochs and this allows the application of renewal-reward theorem.

4.1 The embedded renewal process

In the case of failure the instants of unplanned replacement are regeneration points, and these sequence of regeneration points define a renewal process. A cycle is time between failures and is thus defined by the occurrence of the unplanned replacements after failures.

This structure allows the use of the renewal-reward argument. The average cost rate given the starting state $X(0) = x$ is

$$\mathbb{C}^x(\tau;\ell) = \frac{\mathscr{C}^x(\tau;\ell)}{\ell^x(\tau;\ell)} \tag{2}$$

where $\mathscr{C}^x$ is the expected cost per cycle and ℓ^x is the expected cycle length.

4.2 Expected cost per cycle

The costs incurred in a cycle are random. Let $C^x(\tau;\ell)$ denote the cost per cycle with starting state $X(0) = x$. Upon inspection at τ if the system state $X(\tau)$ is found in $\mathbb{A}_0(\ell)$ the cyclic cost consists of an inspection cost C_0 and future costs $C^{X(\tau)}(\tau;\ell)$ incurred by taking no action. On finding the system state $X(\tau)$ in $\mathbb{A}_1(\ell)$ the cost per cycle is made up from an inspection cost and a replacement cost C_R and future costs $C^0(\tau;\ell)$ incurred by a planned replacement which resets the process $X(\tau)$ to zero. On failure at regeneration point $T_{k:n}(< \tau)$ the costs incurred include an unplanned replacement cost $C_F(> C_R)$. In other words,

$$\begin{aligned} C^x_\tau = (\mathrm{C}_0 + C^{X(\tau)}_\tau)\mathrm{I}\,(X(\tau) \in \mathbb{A}_0(j)) + \left(\mathrm{C}_R + C^0_\tau\right)\mathrm{I}\,(X(\tau) \in \mathbb{A}_1(j)) \\ + \mathrm{C}_F\mathrm{I}\,(X(\tau) \in \mathbb{A}_2) \end{aligned}$$

Taking the expectation follows that

$$\begin{aligned} \mathscr{C}^x(\tau;\ell) = \mathbb{E}\left[C^x(\tau;\ell)\right] = \sum_{r=x}^{\ell-1} (\mathrm{C}_0 + \mathscr{C}^r(\tau;\ell))\,\pi_{xr}(t;\theta) \\ + \left(\mathrm{C_R} + \mathscr{C}^0(\tau;\ell)\right)\sum_{r=\ell_x}^{k-1} \pi_{xr}(\tau;\theta)) + \mathrm{C_F}F_{k-x:n-x}(\tau) \end{aligned} \tag{3}$$

where $\ell_x = \max(\ell, x)$ and $\pi_{xr}(t;\theta)$ denotes the transition probability of X:

$$\pi_{xr}(\tau;\theta) = \binom{n}{r} F^r(\tau)(1 - F(\tau))^{n-r}\,(1 + \theta\,\psi_n(r;\tau))$$

4.3 Expected cycle length

Let $L^x(\tau;\ell)$ denote the cycle length starting from $X(0)=x$. The expected cycle length is obtained with the same argument as the expected cost: on inspection if the system state is observed in $\mathbb{A}_0(\ell)$ the cycle consists of a full period of inspection τ and the remaining cycle length $L^{X(\tau)}(\tau;\ell)$ starting from current state $X(\tau)$. On observing the system in $\mathbb{A}_1(\ell)$ the cycle is made up from a full period τ and the remaining cycle length $L^0(\tau;\ell)$ starting from updated state zero. On failure at $T_{k:n}$ the cycle length becomes complete. In the view of above argument, we have

$$L^x_\tau = \left(\tau + L^{X(\tau)}_\tau\right) I\left(X(\tau)\in\mathbb{A}_0\right) + \left(\tau + L^0_\tau\right) I\left(X(\tau)\in\mathbb{A}_1\right) + T_{n-x:n-x} I\left(X(\tau)\in\mathbb{A}_2\right)$$

$$\ell^x(\tau;\ell) = \mathbb{E}\left[L^x(\tau;\ell)\right] = \sum_{r=x}^{\ell-1}\left(\tau+\ell^x(\tau;\ell)\right)\pi_{xr}(\tau;\theta) + \left(\tau+\ell^0(\tau;\ell)\right)\sum_{r=\ell_x}^{k-1}\pi_{xr}(\tau;\theta)) + \mu_{n-x}(k-x,\tau;\theta) \quad (4)$$

where $\mu_{n-x}(k-x,\tau;\theta)$ denotes the mean time to failure of an $k-x$-out-of-$(n-x)$ system within the first inspection epoch. Our purpose is to determine optimal inspection and threshold-type replacement policy implemented by optimal maintenance parameters $(\tau^*,\ell^*) = argmin_{(\tau,\ell)\in(0,\infty)\times(0,\infty)}\mathbb{C}^x(\tau;\ell)$.

5 Specific models

In this section we show how some models emerge as special cases. They are recovered by the appropriate choice of the threshold parameter ℓ.

5.1 $\langle a, X(t)\rangle = \{a_0, a_2\}$

The repair model with two kinds of repair action: no action (a_0) and corrective replacement (a_2) is recovered if $\ell = k$. In this case the expected cost per cycle

(3) and the expected cycle length (4) respectively become

$$\mathscr{C}^x(\tau;k) = \sum_{r=x}^{k-1} (\mathrm{C}_0 + \mathscr{C}^r(\tau;k))\,\pi_{xr}(t;\theta) + \mathrm{C_F}F_{k-x:n-x}(\tau)$$

and

$$\ell^x(\tau;k) = \sum_{r=x}^{k-1} (\tau + \ell^x(\tau;k))\,\pi_{xr}(\tau;\theta) + \mu_{n-x}(k-x,\tau;\theta).$$

5.2 $\langle a, X(t)\rangle = \{a_1, a_2\}$

The repair actions of the model are restricted to two kinds of action: preventive replacement (a_1) and corrective replacement (a_2) if we set $\ell = x$. In this case the expected cost per cycle (3) and the expected cycle length (4) respectively become

$$\mathscr{C}^x(\tau;x) = \big(\mathrm{C_R} + \mathscr{C}^0(\tau;x)\big) \sum_{r=x}^{k-1} \pi_{xr}(\tau;\theta)) + \mathrm{C_F}F_{k-x:n-x}(\tau)$$

and

$$\ell^x(\tau;x) = \big(\tau + \ell^0(\tau;x)\big) \sum_{r=x}^{k-1} \pi_{xr}(\tau;\theta)) + \mu_{n-x}(k-x,\tau;\theta)$$

6 Main results

For the numerical illustration of the model we consider a 3-out-of-4 system and set $(\alpha,\beta) = (2,2)$ and $(C_0, C_R, C_F) = (0.5, 5, 10)$.

Table 1: Optimal solutions for different θ.

θ	τ^*	ℓ^*	$\mathbb{C}^0(\tau^*,\ell^*)$
0.0[a]	0.493	2	1.985
0.5	0.490	2	1.989
0.9	0.486	2	1.993

[a]Independent components.

Table 1 illustrates optimal solutions for different dependence degrees θ given the starting state $X(0) = 0$. The model suggests that for $\theta = 0.5$, to

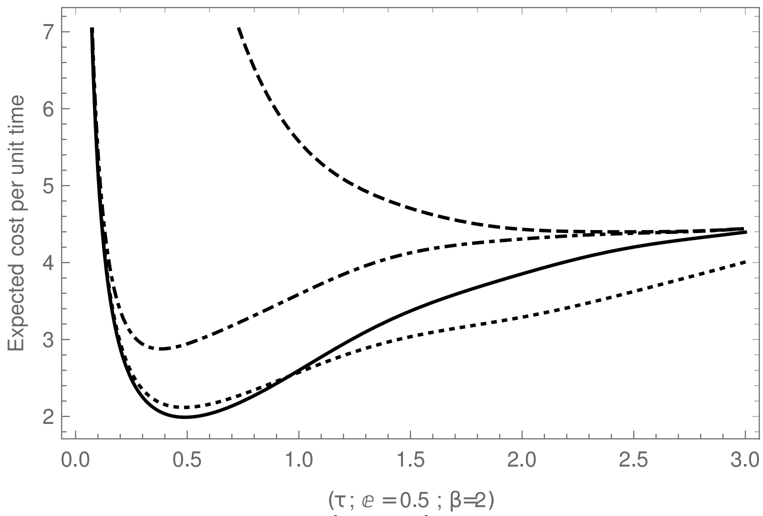

Figure 2: $\mathbb{C}^0(\tau;\ell)$ for different threshold values $\ell \in \{0,1,2,3\}$. The solid line, dotted line, dash-dotted line, dashed line correspond to $\ell = 2$, $\ell = 3$, $\ell = 1$ and $\ell = 0$.

reveal the true state of components and take an appropriate action, inspections should be scheduled with the optimal period of inspection $\tau^* = 0.49$: on inspection if at most one component is found in failed state no action should be taken, otherwise either a preventive replacement should be scheduled if two components are observed in failed state ($\mathbb{A}_1(\ell^*) = \{2\}$), or a corrective replacement has to be carried out upon failure. These policies incur the minimum cost $\mathbb{C}^0(\tau^*,\ell^*) = 1.989$. The results reveal that the optimal replacement threshold ℓ^* is not sensitive to θ, but increasing θ makes inspections more frequent. As noted, the model penalizes a costly strategy which favors too many inspections as dependence degree decreases. Also, an evolution of the average cost rate for different threshold values $\ell \in \{0,1,2,3\}$ and dependence values $\theta \in \{0,0.5\}$ are illustrated by Figure 2 and Figure 3.

Table 2: Optimal solutions for different x.

x	τ^*	ℓ^*	$\mathbb{C}^x(\tau^*,\ell^*)$
0	0.490	2	1.989
1	0.463	2	2.037
2	0.433	2	2.097

The behavior of optimal solutions is examined with respect to the redundancy level of the system (starting state x) for $\theta = 0.5$ (see Table 2). The

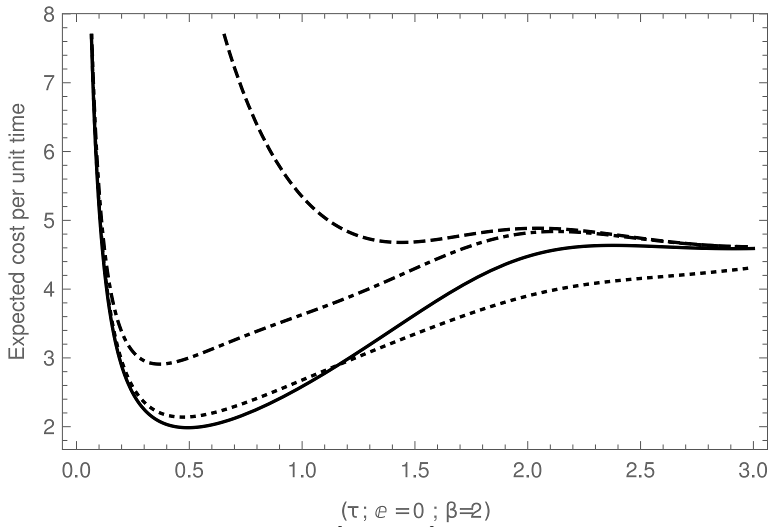

Figure 3: $\mathbb{C}^0(\tau;\ell)$ for different threshold values $\ell \in \{0,1,2,3\}$. The solid line, dotted line, dash-dotted line, dashed line correspond to $\ell = 2$, $\ell = 3$, $\ell = 1$ and $\ell = 0$.

results reveal that the optimal replacement threshold is not sensitive to x, but decreasing redundancy level results in an increase in both inspection frequency and maintenance cost.

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Tests of Exponentiality for the Progressively Type-II Censored Sample: new test and comparative study

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Abstract: A new divergences are defined by using Tsallis divergence and a measure of discrepancy between equilibriums associated with two distributions is proposed. Then utilizing the progressively Type-II censored sample, we construct goodness of fit tests for exponentiality based on the estimation of proposed divergences. To investigate the performance of the mentioned tests, Monte Carlo simulations are performed. The powers of the proposed tests are then compared with other existing tests. Finally, an example is used of the proposed tests.

Keywords: Cumulative residual Tsallis divergence, Monte Carlo simulation, Progressively Type-II censored sample.

1 Introduction

Because of the importance of exponential distribution in reliability and lifetime models, many tests with complete samples and some procedures under censored data have been presented in previous studies, attempting to determine the appropriateness of an exponential model for a given dataset. Type-I and Type-II censoring schemes are the most popular ones among the different censoring schemes. One of the disadvantages of these censoring schemes is

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the impossibility to withdraw units during the experiment, so a generalization of the classical Type-II censoring scheme, known as the progressive Type-II censoring scheme (PCS Type-II), was proposed by analysts to pull back units amid the experiment.

There are several goodness of fit tests available in the literature based on censored data for exponentiality test. [15] introduced a test statistic for the exponential distribution and obtained the exact distribution of the test statistic under the null hypothesis. [8] extended the goodness of fit test based on Kullback–Leibler (KL) information for PCS Type-II data for three distributions. [10] developed some goodness of fit tests for the exponential distribution based on Type-I censored samples. Recently; [11] generalized the cumulative residual KL (CRKL) information to the censored case and used the estimate of modified version of CRKL as a goodness of fit test statistic with the Type-II censored data. [2] developed a general goodness of fit test for Type-II censored data by using a new estimate of KL information for Type-II censored data. Also, they considered testing for exponentiality under Type-II censored data as a special case of this general test. [5] constructed two goodness of fit tests based on the CRKL and cumulative KL (CKL) information for testing exponentiality with PCS Type-II data. [1] introduced a general goodness of fit test for PCS Type-II data based on a new estimate of KL information and then used the proposed test statistic for testing exponentially based on PCS Type-II data.

Let X be a non negative absolutely continuous random variable having cumulative distribution function (cdf) F, and the probability density function (pdf) f. [13] introduced a new uncertainty measure, the cumulative residual entropy (CRE), which for a non negative random variable X is defined as follows:

$$CRE(F) = -\int_0^\infty \bar{F}(x)\ln \bar{F}(x)dx.$$

Similar to the CRE, [7] proposed the cumulative entropy (CE) through

$$CE(F) = -\int_0^\infty F(x)\ln F(x)dx.$$

Consider two nonnegative and absolutely continuous random variables X and Y with pdfs f and g, cdfs F and G, respectively. Then, the KL informations as a measure of discrepancy between f and g is given by

$$KL(f:g) = \int_0^\infty f(x) \ln \frac{f(x)}{g(x)} dx,$$

and the Tsallis divergence between f and g is defined as (see Tsallis, 1988)

$$D_T(f,g) = \frac{1}{\alpha - 1}\left[\int_0^\infty f^\alpha(x) g^{1-\alpha}(x) dx - 1\right], \quad \alpha(\neq 1) > 0. \tag{1}$$

[5] suggested an extension of the KL information to the survival function, which is CRKL, as follows:

$$CRKL(F:G) = \int_0^\infty \bar{F}(x) \ln \frac{\bar{F}(x)}{\bar{G}(x)} dx - [E(X) - E(Y)],$$

where $\bar{F}(x)$ and $\bar{G}(x)$ are the survival functions of random variables X and Y, respectively. [12] considered another extension to the cumulative distribution, which is called CKL and defined as follows:

$$CKL(F:G) = \int_0^\infty F(x) \ln \frac{F(x)}{G(x)} dx - [E(Y) - E(X)].$$

Let $w(t)$ be a non-negative function, so that $0 < E(w(t)) < \infty$, then we can define the weighted random variable X^* with density function

$$f^*(t) = \frac{w(t) f_X(t)}{E(w(X))}, \quad t \geq 0. \tag{2}$$

The equilibrium distribution results as a special case when $w(t) = \frac{1}{r_X(t)}$, where $r_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}$ is failure rate function of X; then X^* is said the equilibrium random variable associated with X. The pdf of X^* is $f^*(t) = \frac{\bar{F}_X(t)}{E(X)}$.

Let f^* and g^* be the equilibrium pdfs respectively associated with f and g. Then, we define the Tsallis divergence based on equilibrium distributions as follows

$$D_T(f^*, g^*) = \frac{1}{\alpha - 1}\left[\frac{E^{\alpha-1}(Y)}{E^\alpha(X)} \int_0^\infty \bar{F}^\alpha(x) \bar{G}^{1-\alpha}(x) dx - 1\right], \quad \alpha(\neq 1) > 0. \tag{3}$$

The testing of interest in this article, is

$$H_0: F(x) = F_0(x) \quad vs \quad H_1: F(x) \neq F_0(x),$$

where $F_0(x) = 1 - \exp(-\frac{x}{\theta})$ with $x > 0,\ \theta > 0$, and θ is the unknown parameter.
The performance of the proposed tests is compared to proposed tests of [4] and [1] for PCS Type-II data.

2 Extentions of Tsallis divergence and test statistics

In this section, the new measures of distance between two distributions that are similar to Tsallis divergences are defined.

Definition 2.1. Let X and Y be two non negative and absolutely continuous random variables with cdfs F and G and pdfs f and g, respectively. Then cumulative residual Tsallis (CRT) and cumulative Tsallis (CT) divergence between these distributions are respectively as follows

$$CRT(F:G) = \frac{1}{\alpha-1}\left[\int_0^\infty \bar{F}^\alpha(x)\bar{G}^{1-\alpha}(x)dx - \alpha E(X) - (1-\alpha)E(Y)\right], \quad 0<\alpha<1. \tag{4}$$

$$\begin{aligned} CT(F:G) &= \frac{1}{\alpha-1}\left[\int_0^\infty F^\alpha(x)G^{1-\alpha}(x)dx \right.\\ &\quad \left. - \alpha\int_0^\infty F(x)dx - (1-\alpha)\int_0^\infty G(x)dx\right], \quad 0<\alpha<1. \end{aligned} \tag{5}$$

Lemma 2.2. *CRT(F:G)$\geq$ 0 and equality holds iff $F = G$.*

Proof. By applying the Hölder inequality, we obtain

$$\int_0^\infty \bar{F}^\alpha(x)\bar{G}^{1-\alpha}(x)dx \leq \left(\int_0^\infty \bar{F}(x)dx\right)^\alpha \left(\int_0^\infty \bar{G}(x)dx\right)^{1-\alpha}, \quad 0<\alpha<1, \tag{6}$$

and by using the Young inequality, we get

$$\left(\int_0^\infty \bar{F}(x)dx\right)^\alpha \left(\int_0^\infty \bar{G}(x)dx\right)^{1-\alpha} \leq \alpha\int_0^\infty \bar{F}(x)dx + (1-\alpha)\int_0^\infty \bar{G}(x)dx. \tag{7}$$

Therefore, by (2) and (3) and dividing by $\alpha - 1$, the desired inequality follows. In the Hölder inequality, equality holds iff $\bar{F} = c\bar{G}$ (c is a positive constant) and in the Young inequality, equality holds iff $\int_0^\infty \bar{F}(x)dx = \int_0^\infty \bar{G}(x)dx$. Thus, $c = 1$

and

$CRT(F:G) = 0$ iff $F = G$. □

Lemma 2.3. *CT(F:G)≥ 0 and equality holds iff* $F = G$.

Proof. The proof is similar to the Lemma 2.2. □

Remark 2.4. Note that $\lim_{\alpha\to 1} CRT = CRKL$ and $\lim_{\alpha\to 1} CT = CKL$.

2.1 Testing procedures based on the extentions of Tsallis divergence

In this section, test statistics are constructed for testing exponentiality with the PCS Type-II data and then some competing tests are considered to be compared with the mentioned tests. Accordingly, letting $F(x) = F_{m:n}(x)$ (the cdf estimator based on progressively Type-II right censored data) and $G(x) = F_0(x)$ in (1), we have

$$\begin{aligned} CRT(F_{m:n}:F_0) &= -\frac{\theta}{(\alpha-1)^2}\left[\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}\left(e^{-\frac{x_{i:m:n}}{\theta}(1-\alpha)}-e^{-\frac{x_{i+1:m:n}}{\theta}(1-\alpha)}\right)\right] \\ &- \frac{\alpha}{\alpha-1}\left[\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})\right] \\ &+ \theta\left(1-e^{-\frac{x_{m:m:n}}{\theta}}\right), \end{aligned} \tag{8}$$

where $\alpha_{0:m:n} = x_{0:m:n} = 0$ and $\alpha_{i:m:n}$ is the expected value of the Type-II progressively censored order statistic from the uniform distribution on (0,1), which is given by [3]. Dividing (7) by $\int_0^{x_{m:m:n}}(1-F_{m:n}(x))dx$, we obtain the proposed test as follows:

$$\begin{aligned} CRT_{mn} &= -\frac{\hat{\theta}}{(\alpha-1)^2}\left[\frac{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}\left(e^{-\frac{x_{i:m:n}}{\hat{\theta}}(1-\alpha)}-e^{-\frac{x_{i+1:m:n}}{\hat{\theta}}(1-\alpha)}\right)}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})}\right] \\ &+ \frac{\hat{\theta}\left(1-e^{-\frac{x_{m:m:n}}{\hat{\theta}}}\right)}{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})} - \frac{\alpha}{\alpha-1}, \end{aligned} \tag{9}$$

where $\hat{\theta} = \frac{1}{m}\sum_{i=1}^{m}(R_i+1)x_{i:m:n}$ is the maximum likelihood estimate (MLE) based on the PCS Type-II sample.

Similarly, for (2), we have

$$\begin{aligned} CT(F_{m:n}:F_0) &= \frac{1}{(\alpha-1)}\left[\sum_{i=1}^{m-1}(\alpha_{i:m:n})^{\alpha}\int_{x_{i:m:n}}^{x_{i+1:m:n}}(1-e^{-\frac{x(1-\alpha)}{\theta}})dx\right] \\ &- \frac{\alpha}{\alpha-1}\left[\sum_{i=1}^{m-1}(\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})\right] \\ &+ \int_0^{x_{m:m:n}}(1-e^{-\frac{x}{\theta}})dx. \end{aligned} \tag{10}$$

Dividing (10) by $\int_0^{x_{m:m:n}} F_{m:n}(x)dx$, we obtain the proposed test as follows:

$$\begin{aligned} CT_{mn} &= \frac{1}{(\alpha-1)}\left[\frac{\sum_{i=1}^{m-1}(\alpha_{i:m:n})^{\alpha}\int_{x_{i:m:n}}^{x_{i+1:m:n}}(1-e^{-\frac{x(1-\alpha)}{\hat{\theta}}})dx}{\sum_{i=1}^{m-1}(\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})}\right] \\ &+ \frac{\int_0^{x_{m:m:n}}(1-e^{-\frac{x}{\hat{\theta}}})dx}{\sum_{i=1}^{m-1}(\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n})} - \frac{\alpha}{\alpha-1}, \end{aligned} \tag{11}$$

where $\hat{\theta}$ is the MLE of based on the PCS Type-II sample.

2.2 Testing procedures based on equilibrium distributions for the Tsallis divergences

Similar using (3) based on the PCS Type-II data, we obtain the proposed test as follow:

$$\begin{aligned} D^*_{T_{mn}} &= \frac{1}{\alpha-1}\left[\frac{\hat{\theta}^{\alpha-1}\int_0^{x_{m:m:n}}(1-F_{m:n}(x))^{\alpha}e^{-\frac{x}{\hat{\theta}}(1-\alpha)}dx}{(\int_0^{x_{m:m:n}}(1-F_{m:n}(x))dx)^{\alpha}}-1\right] \\ &= -\frac{\hat{\theta}^{\alpha}}{(\alpha-1)^2}\left[\frac{\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})^{\alpha}(e^{-\frac{x_{i:m:n}}{\hat{\theta}}(1-\alpha)}-e^{-\frac{x_{i+1:m:n}}{\hat{\theta}}(1-\alpha)})}{(\sum_{i=0}^{m-1}(1-\alpha_{i:m:n})(x_{i+1:m:n}-x_{i:m:n}))^{\alpha}}\right] \\ &- \frac{1}{\alpha-1}, \end{aligned} \tag{12}$$

where $\alpha_{0:m:n} = x_{0:m:n} = 0$ and $\hat{\theta}$ is the MLE of the PCS Type-II sample. Note that all the three proposed test statistics are scale-invariant.

3 Simulation study

For large values of the proposed test statistics, the null hypothesis will be rejected. The power values of the proposed tests depend on two things, the α values and type of failure rate function of alternatives. Thus, the alternatives are selected according to the type of failure rate function as follows:

- Increasing failure rate (IFR): Gamma and Weibull (shape parameter 2),
- Decreasing failure rate (DFR): Gamma and Weibull (shape parameter 0.5),
- Non-monotone failure rate (NFR): Log-normal (shape parameter 0.5), Log-normal (shape parameter 1).

Since the α values have an important role in determining the power of the proposed tests, then the α value that maximizes the power, is considered according to the type of failure rate function. Moreover since the CT_{mn} and CRT_{mn} statistics have not good performance, respectively, for alternatives with IFR and DFR functions, thus the CT_{mn} and CRT_{mn} statistics are recommended for DFR function and IFR function, respectively. The α value, for $D_{T^*_{mn}}$ and CRT_{mn} statistics, when the alternatives have the IFR function, is suggested to be 2 and 0.01, respectively and for $D^*_{T_{mn}}$ and CT_{mn} statistics, when the alternatives have the DFR function, is suggested to be 0.01. For alternatives with NFR functions, the α value is suggested 0.01 for the proposed statistics.
The power values are determined for the 27 censoring schemes used by [9]. These censoring schemes are given in Table 1. To obtain the power values 50,000 random samples for several sample sizes and PCS, are generated. Also by following [1], the values of w which maximize the power of the TA statistic are chosen.
Tables 2–4 present power values of the proposed tests and the existing tests at the significance level 0.10 based on the type of failure rate function. In these tables, the proposed test of [4]. is denoted by T. According to these tables, it can be said that the proposed tests are evidency consistent because with increasing sample size, the test power close to 1. Table 2 (for alternatives with

IFR functions) indicates that, almost in the most cases, the TA statistic has higher power than other tests. Also, we can see that, the difference of powers of the CRT_{mn} and TA statistics do not differ much. Although the TA statistics have good powers. One of the disadvantages of this statistic is that we should calculate the power values for three different values of window size w, and, for different censoring schemes, there is not a window size w of same value. While if [4] had considered w values proposed by [1] for each censorship plan, they would have had higher powers compared to TA. Since usually the CRT_{mn} statistic for the scheme $R=(n-m,0,\ldots,0)$ shows higher power than the other schemes, so this statistic for early censoring is recommended.
In Table 3 for alternatives with DER functions, the CT_{mn} statistic has higher power than $D^*_{T_{mn}}$ and the other existing tests except in the censoring scheme 24. In this table, the scheme $R=(n-m,0,\ldots,0)$ generally indicates better power than the other schemes. It can be concluded that early censoring situations seem to possess higher power. Therefore for alternatives with DER functions, the use of CT_{mn} statistic for the case of early censoring is recommended. Table 4 shows that the TA and CRT_{mn} statistics have approximately higher powers than the other tests, but for different censoring schemes a general conclusion cannot be suggested.

4 Illustrative example

In this section, the proposed tests procedure are investigated by an example. In this example, the real dataset with $n=19$ and $m=8$ is considered, Nelson (1982) reported data on times to breakdown of an in-sulating fluid in an accelerated test which was done at different test voltages. From these data, [14] produced a PCS Type-II sample of size from observations which was recorded at 34 kilovolts. These progressively censored data are given in Table 5. Table 6 indicates the critical values and test statistics. Based on Table 6, all of the tests at the significance level 0.10, show that this progressively Type-II right censored sample comes from an exponential distribution.

Table 1: Progressive censoring schemes used in the Monte Carlo simulations

Scheme No.	n	m	$(R_1,\ldots,R_m)$
[1]	20	8	$R_1=12, R_i=0$ for $i\neq 1$
[2]			$R_8=12, R_i=0$ for $i\neq 8$
[3]			$R_1=R_8=6, R_i=0$ for $i\neq 1,8$
[4]		12	$R_1=8, R_i=0$ for $i\neq 1$
[5]			$R_{12}=8, R_i=0$ for $i\neq 12$
[6]			$R_3=R_5=R_7=R_9=2, R_i=0$ for $i\neq 3,5,7,9$
[7]		16	$R_1=4, R_i=0$ for $i\neq 1$
[8]			$R_{16}=4, R_i=0$ for $i\neq 16$
[9]			$R_5=4, R_i=0$ for $i\neq 5$
[10]	40	10	$R_1=30, R_i=0$ for $i\neq 1$
[11]			$R_{10}=30, R_i=0$ for $i\neq 10$
[12]			$R_1=R_5=R_{10}=10, R_i=0$ for $i\neq 1,5,10$
[13]		20	$R_1=20, R_i=0$ for $i\neq 1$
[14]			$R_{20}=20, R_i=0$ for $i\neq 20$
[15]			$R_i=1$ for $i=1,2,\ldots,20$
[16]		30	$R_1=10, R_i=0$ for $i\neq 1$
[17]			$R_{30}=10, R_i=0$ for $i\neq 30$
[18]			$R_1=R_{30}=5, R_i=0$ for $i\neq 1,30$
[19]	60	20	$R_1=40, R_i=0$ for $i\neq 1$
[20]			$R_{20}=40, R_i=0$ for $i\neq 20$
[21]			$R_1=R_{20}=10, R_{10}=20, R_i=0$ for $i\neq 1,10,20$
[22]		40	$R_1=20, R_i=0$ for $i\neq 1$
[23]			$R_{40}=20, R_i=0$ for $i\neq 40$
[24]			$R_{2i-1}=1, R_{2i}=0$ for $i=1,2,\ldots,20$
[25]		50	$R_1=10, R_i=0$ for $i\neq 1$
[26]			$R_{50}=10, R_i=0$ for $i\neq 50$
[27]			$R_1=R_{50}=5, R_i=0$ for $i\neq 1,50$

Table 2: Power of the proposed tests for the alternatives with the IFR function at the significance level 0.10 for several schemes.

	W(2)				G(2)			
Scheme No.	$D^{*}_{T_{mn}}$	CRT_{mn}	T	TA	$D^{*}_{T_{mn}}$	CRT_{mn}	T	TA
[1]	0.858	0.893	0.892	**0.896**	0.610	0.614	**0.648**	0.231
[2]	**0.643**	0.661	0.634	0.627	**0.488**	0.457	0.459	0.486
[3]	**0.776**	0.748	0.725	0.712	**0.549**	0.513	0.512	0.454
[4]	0.879	0.938	0.916	**0.949**	0.684	0.613	**0.722**	0.484
[5]	**0.811**	0.803	0.783	0.799	**0.612**	0.533	0.583	0.603
[6]	0.807	0.898	0.891	**0.949**	0.633	0.573	**0.675**	0.226
[7]	0.896	0.959	0.958	**0.976**	0.714	0.658	**0.765**	0.651
[8]	0.900	0.903	**0.922**	0.920	**0.691**	0.658	0.686	0.688
[9]	0.914	0.958	0.970	**0.983**	0.702	0.680	**0.772**	0.633
[10]	0.939	0.942	0.953	**0.965**	0.666	0.756	**0.814**	0.232
[11]	**0.726**	0.716	0.678	0.683	0.594	0.538	0.550	**0.597**
[12]	0.817	**0.841**	0.812	0.815	**0.684**	0.634	0.646	0.527
[13]	0.963	0.988	0.988	**0.995**	0.765	0.791	**0.899**	0.670
[14]	**0.945**	0.933	0.918	0.922	0.794	0.724	0.759	**0.806**
[15]	0.905	0.963	0.970	**0.991**	0.768	0.756	**0.862**	0.465
[16]	0.977	0.996	0.998	**1.000**	0.800	0.827	**0.942**	0.882
[17]	**0.992**	0.994	0.991	0.990	0.892	0.842	0.879	**0.904**
[18]	0.993	**0.997**	0.996	0.995	0.900	0.863	**0.902**	0.901
[19]	0.979	0.989	0.993	**0.998**	0.756	0.850	**0.934**	0.591
[20]	**0.941**	0.930	0.898	0.901	0.825	0.747	0.776	**0.834**
[21]	**0.980**	0.978	0.970	0.974	**0.857**	0.816	0.851	0.738
[22]	0.992	0.999	**1.000**	**1.000**	0.815	0.906	**0.980**	0.945
[23]	0.998	**0.999**	0.998	0.996	0.957	0.927	0.933	**0.965**
[24]	0.964	0.992	0.999	**1.000**	0.773	0.871	**0.960**	0.849
[25]	0.994	0.999	**1.000**	**1.000**	0.831	0.924	**0.991**	0.981
[26]	**1.000**	**1.000**	**1.000**	**1.000**	0.971	0.956	0.973	**0.985**
[27]	**1.000**	**1.000**	**1.000**	**1.000**	0.960	0.961	0.979	**0.984**

For each censoring scheme, the greatest powers are in bold.

Table 3: Power of the proposed tests for the alternatives with the DFR function at the significance level 0.10 for several schemes.

	W(0.5)				G(0.5)			
Scheme No.	$D^*_{T_{mn}}$	CT_{mn}	T	TA	$D^*_{T_{mn}}$	CT_{mn}	T	TA
[1]	0.661	**0.717**	0.064	0.303	0.420	**0.467**	0.014	0.161
[2]	0.005	**0.291**	0.069	0.213	0.010	**0.219**	0.047	0.166
[3]	0.002	**0.448**	0.073	0.252	0.005	**0.323**	0.040	0.177
[4]	0.750	**0.806**	0.341	0.476	0.455	**0.511**	0.097	0.230
[5]	0.002	**0.590**	0.272	0.387	0.006	**0.398**	0.134	0.250
[6]	0.752	**0.792**	0.344	0.489	0.496	**0.546**	0.109	0.258
[7]	0.812	**0.869**	0.416	0.605	0.471	**0.544**	0.104	0.291
[8]	0.069	**0.847**	0.402	0.564	0.016	**0.613**	0.147	0.318
[9]	0.830	**0.889**	0.437	0.637	0.509	**0.587**	0.111	0.315
[10]	0.728	**0.787**	0.217	0.442	0.448	**0.508**	0.050	0.227
[11]	0.003	**0.345**	0.152	0.270	0.005	**0.278**	0.112	0.226
[12]	0.003	**0.672**	0.187	0.381	0.005	**0.567**	0.111	0.287
[13]	0.866	**0.919**	0.661	0.748	0.530	**0.615**	0.234	0.395
[14]	0.000	**0.760**	0.543	0.600	0.001	**0.570**	0.336	0.421
[15]	0.628	**0.864**	0.627	0.756	0.336	**0.636**	0.309	0.485
[16]	0.934	**0.971**	0.865	0.892	0.583	**0.695**	0.381	0.520
[17]	0.000	**0.959**	0.829	0.830	0.001	**0.779**	0.487	0.551
[18]	0.415	**0.979**	0.854	0.866	0.079	**0.821**	0.460	0.550
[19]	0.866	**0.922**	0.673	0.765	0.539	**0.630**	0.246	0.429
[20]	0.000	**0.726**	0.524	0.566	0.001	**0.586**	0.375	0.450
[21]	0.000	**0.916**	0.677	0.745	0.001	**0.795**	0.429	0.553
[22]	0.965	**0.990**	0.949	0.960	0.638	**0.769**	0.519	0.639
[23]	0.000	**0.980**	0.928	0.904	0.000	**0.857**	0.670	0.677
[24]	0.898	0.947	0.924	**0.952**	0.572	0.665	0.570	**0.700**
[25]	0.983	**0.997**	0.989	0.985	0.682	**0.821**	0.680	0.711
[26]	0.178	**0.997**	0.986	0.969	0.007	**0.928**	0.783	0.742
[27]	0.918	**0.999**	0.990	0.978	0.402	**0.934**	0.765	0.741

For each censoring scheme, the greatest powers are in bold.

Table 4: Power of the proposed tests for the alternatives with the NFR function at the significance level 0.10 for several schemes.

	L(0,0.5)					L(0,1)				
S.N	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA
[1]	0.254	0.990	0.000	0.995	0.999	0.134	0.280	0.123	0.304	0.364
[2]	0.956	0.993	0.938	0.987	0.987	0.348	0.468	0.287	0.428	0.440
[3]	0.993	0.998	0.453	0.995	0.996	0.410	0.469	0.082	0.437	0.459
[4]	0.008	0.996	0.001	0.996	1.000	0.194	0.269	0.189	0.285	0.353
[5]	0.975	1.000	0.955	0.997	0.999	0.285	0.497	0.185	0.415	0.484
[6]	0.022	0.994	0.000	0.991	0.998	0.164	0.316	0.157	0.283	0.310
[7]	0.001	0.998	0.001	0.997	0.999	0.241	0.250	0.240	0.260	0.351
[8]	0.967	1.000	0.866	1.000	1.000	0.175	0.451	0.084	0.399	0.486
[9]	0.001	0.997	0.001	0.998	1.000	0.236	0.262	0.232	0.269	0.336
[10]	0.615	0.996	0.037	1.000	1.000	0.137	0.283	0.123	0.483	0.581
[11]	0.996	1.000	0.995	0.999	0.999	0.561	0.701	0.492	0.614	0.652
[12]	0.999	1.000	0.295	1.000	1.000	0.513	0.691	0.021	0.635	0.653
[13]	0.016	0.999	0.052	1.000	1.000	0.255	0.225	0.256	0.440	0.588
[14]	1.000	1.000	1.000	1.000	1.000	0.478	0.784	0.379	0.667	0.758
[15]	0.937	1.000	0.000	1.000	1.000	0.201	0.500	0.093	0.520	0.618
[16]	0.001	0.999	0.056	1.000	1.000	0.340	0.181	0.343	0.414	0.588
[17]	1.000	1.000	1.000	1.000	1.000	0.250	0.708	0.146	0.648	0.758
[18]	0.999	1.000	0.985	1.000	1.000	0.170	0.569	0.107	0.619	0.743
[19]	0.251	0.999	0.364	1.000	1.000	0.227	0.213	0.230	0.579	0.736
[20]	1.000	1.000	1.000	1.000	1.000	0.695	0.897	0.620	0.801	0.858
[21]	1.000	1.000	0.698	1.000	1.000	0.509	0.799	0.028	0.779	0.793
[22]	0.002	0.999	0.408	1.000	1.000	0.390	0.137	0.403	0.540	0.764
[23]	1.000	1.000	1.000	1.000	1.000	0.411	0.863	0.323	0.837	0.894
[24]	0.004	0.999	0.002	1.000	1.000	0.243	0.245	0.245	0.496	0.751
[25]	0.001	0.999	0.474	1.000	1.000	0.449	0.113	0.464	0.572	0.773
[26]	1.000	1.000	1.000	1.000	1.000	0.178	0.672	0.153	0.782	0.880
[27]	1.000	1.000	1.000	1.000	1.000	0.106	0.462	0.190	0.762	0.878

Table 5: Progressively censored sample generated from the times to breakdown data on insulating fluids tested at 34 kilovolts, given by Viveros and Balakrishnan (1994).

i	1	2	3	4	5	6	7	8
$x_{i:8:19}$	0.19	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	0	0	3	0	3	0	0	5

Table 6: Test statistics and critical values of the tests.

	$D^*_{T_{mn}}$	CRT_{mn}	CT_{mn}	T	TA		
					$w=1$	$w=2$	$w=4$
Test statistic	0.8185	0.0225	-0.0100	-0.0906	-0.0135	-0.0598	-0.1227
Critical value	1.1615	0.0706	-0.0099	0.0662	0.2095	0.0975	0.0579

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Some New Results on Likelihood Ratio Ordering of Spacings of Generalized Order Statistics with Applications

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Abstract: The spacings of ordered random variables appear in many branches of statistical theory and applications such as reliability and life testing. In this article, we study the likelihood ratio ordering of p-spacings of generalized order statistics and establish some more flexible and applicable results. We also settle certain open problems in this regard by providing some useful lemmas. Finally, two applications of these results are indicated in sequential k-out-of-n systems and progressive Type-II censored order statistics.

Keywords: Logconcavity and logconvexity, Progressive Type-II censored order statistics, Sequential order statistics, Stochastic orderings, Total positivity.

1 Introduction

The concept of generalized order statistics (GOS) introduced by [10, 11] as a general framework for models of ordered random variables. Let X be a nonnegative random variable with cumulative distribution function (cdf) F, survival function (sf) $\overline{F} = 1 - F$, and probability density function (pdf) f. Let $h = f/\bar{F}$ be the hazard rate function of X. Random variables $X_{(r,n,\widetilde{m}_n,k)}$,

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$r = 1,2,...,n$, are called GOS if their joint density function is given by

$$\mathbf{f}(x_1,...,x_n) = k\Big(\prod_{j=1}^{n-1}\gamma_{(j,n,\widetilde{m}_n,k)}\Big)\Big(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i}f(x_i)\Big)[\bar{F}(x_n)]^{k-1}f(x_n),$$

for all $F^{-1}(0) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1-)$, where $n \in \mathbb{N}$, $k > 0$ and $m_1,...,m_{n-1} \in \mathbb{R}$ are such that $\gamma_{(r,n,\widetilde{m}_n,k)} = k+n-r+\sum_{j=r}^{n-1} m_j > 0$ for all $r \in \{1,...,n-1\}$, and $\widetilde{m}_n = (m_1,...,m_{n-1})$, if $n \geq 2$ ($\widetilde{m}_n \in \mathbb{R}$ is arbitrary, if $n = 1$). For example, if $m_1 = ... = m_{n-1} = 0$ and $k = 1$, or $m_1 = ... = m_{n-1} = -1$ and $k \in \mathbb{N}$, then the GOS would convert to the order statistics and k-record values, respectively (see Table 1 [10] for complete information of submodels). We denote the general spacings of GOS by $D_{(r,s,n,\widetilde{m}_n,k)} = X_{(s,n,\widetilde{m}_n,k)} - X_{(r-1,n,\widetilde{m}_n,k)}$, with $X_{(0,n,\widetilde{m}_n,k)} \equiv 0$. For $s = r$ and $s = r+p-1$, it is simple spacings and p-spacings (denoted by $D^{(p)}_{(r,n,\widetilde{m}_n,k)}$), respectively.

One of the most important stochastic ordering is the likelihood ratio ordering that implies some other stochastic orderings. We say that X is smaller than Y (with pdf g) in likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x. Also, X or F is said to be ILR (increasing likelihood ratio) [DLR (decreasing likelihood ratio)] if its pdf exists and is logconcave [logconvex] (cf. [14]).

Now, consider the following problems:

(P_1) $X \in DLR \Rightarrow D^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{lr} D^{(p)}_{(r+1,n,\widetilde{m}_n,k)}$;

(P_2) $X \in DLR \Rightarrow D^{(p)}_{(r,n+1,\widetilde{m}_n,k)} \leq_{lr} D^{(p)}_{(r,n,\widetilde{m}_n,k)}$;

(P_3) $X \in DLR \Rightarrow D^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{lr} D^{(p)}_{(r+1,n+1,\widetilde{m}_n,k)}$;

(P_4) $X \in DLR \Rightarrow D^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{lr} D^{(p)}_{(r',n',\widetilde{m}_n,k)}$, $r \leq r'$, $n'-r' \leq n-r$;

(P_5) $X \in ILR \Rightarrow D^{(p)}_{(r,n,\widetilde{m}_n,k)} \geq_{lr} D^{(p)}_{(r+1,n+1,\widetilde{m}_n,k)}$;

(P_6) $X \in ILR \Rightarrow D^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{lr} D^{(p+1)}_{(r-1,n,\widetilde{m}_n,k)}$;

(P_7) $X \in ILR \Rightarrow D^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{lr} D^{(p')}_{(r',n,\widetilde{m}_n,k)}$, $p+1 \leq p'$, $r' \leq r-1$, $p+r = p'+r'$.

For order statistics, [13] obtained (P_1) and (P_2). [9] proved (P_3) (and (P_4) as a corollary of (P_1), (P_2) and (P_3)), (P_5) and (P_6). For GOS, [8] obtained (P_1)-(P_6) under the strong condition $m_1 = ... = m_{n-1}$ in which the marginal and joint pdf of GOS have the closed form representation in this case. Finally, [16] proved (P_1)-(P_4) without the condition $m_1 = ... = m_{n-1}$ using some conditionally results.

In this article, we first give some preliminaries in Section 2. In Section 3, we obtain our main results for very flexible case of different parameters $\widetilde{m}_n$ and $\widetilde{m}'_{n'}$. This enables us to compare the submodels of GOS among themselves, and, more generally, among different submodels (we refer the reader to the recent monograph of [4] for dispersive and star orderings of GOS with different parameters $\widetilde{m}_n$ and $\widetilde{m}'_{n'}$). We extend (P_1)-(P_4) in the unifying Theorem 1 for different m_i and m'_i. We note that (P_5) and (P_6) remained as the open problems for unequal m_i. We extend (P_5) for different m_i and m'_i among simple spacings, i.e., for $p = 1$ in Theorem 2. Also, we extend it for $m'_i = m_i$, but unequal m_i, and for arbitrary p-spacings in Theorem 3.3. We extend (P_7) (which is more general than (P_6)) for different parameters m_i and m'_i in Theorem 4.4. Finally, in Section 4, we explain how these results can be applied in sequential k-out-of-n systems and progressive Type-II censored order statistics.

Throughout the paper, the word increasing (decreasing) is used for non-decreasing (non-increasing) and all expectations are implicitly assumed to exist whenever they are written. Also, we omit some proofs for the sake of brevity.

2 Preliminaries

For the marginal pdf of GOS, [7] obtained the expression

$$f_{X_{(r,n,\widetilde{m}_n,k)}}(x) = c_{r-1}[\bar{F}(x)]^{\gamma_{(r,n,\widetilde{m}_n,k)}-1} g_r(F(x)) f(x),\ x \in \mathbb{R}, \quad (1)$$

where $c_{r-1} = \prod_{i=1}^{r} \gamma_{(i,n,\widetilde{m}_n,k)}$, $r = 1,\dots,n$, $\gamma_{(n,n,\widetilde{m}_n,k)} = k$, and g_r is a particular Meijer's G-function. [15] rediscovered this representation by presenting an integral representation for g_r. For the joint pdf of $X_{(r,n,\tilde{m}_n,k)}$ and $X_{(s,n,\tilde{m}_n,k)}$, $1 \le r < s \le n$, [15] established the expression

$$f_{X_{(r,n,\tilde{m}_n,k)},X_{(s,n,\tilde{m}_n,k)}}(x,y) = c_{s-1}[\bar{F}(x)]^{\gamma_{(r,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)}-1} g_r(F(x)) \\ \times [\bar{F}(y)]^{\gamma_{(s,n,\tilde{m}_n,k)}-1} \psi_{s-r-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right) f(x)f(y),\ x<y, \quad (2)$$

where $\psi_0(t)=1$, $\psi_1(t)=\delta_{m_{r+1}}(1-t)$,

$$\psi_\alpha(t) = \int_t^1 \int_{u_{\alpha-1}}^1 \dots \int_{u_2}^1 \delta_{m_{r+1}}(1-u_1) \prod_{i=1}^{\alpha-1} u_i^{m_{r+i+1}} du_1 \dots du_{\alpha-2} du_{\alpha-1},\ 0 \le t \le 1,$$

for $\alpha = 2,3,\dots$, and $\delta_m(t) = \begin{cases} \frac{1}{m+1}(1-(1-t)^{m+1}), & m \neq -1 \\ -\ln(1-t), & m=-1 \end{cases}$, $t \in (0,1)$.

[1] and [2, 3] studied logconcavity properties of the function g_r and GOS. According to Lemmas 2.1 and 3.1 of [1], we have

$$g_1(t)=1,\ g_r(t) = \int_0^t g_{r-1}(u)[1-u]^{m_{r-1}} du,\ 0 \le t \le 1,\ r=2,\dots,n, \quad (3)$$

$$\psi_0(t)=1,\ \psi_\alpha(t) = \int_t^1 \psi_{\alpha-1}(u) u^{m_{r+\alpha}} du,\ 0 \le t \le 1,\ \alpha = 1,2,\dots\,. \quad (4)$$

Now, substituting r with $r-1$ in (2) and some calculations, for $2 \le r \le s \le n$, we obtain

$$f_{D_{(r,s,n,\tilde{m}_n,k)}}(x) = c_{s-1} \int_0^{+\infty} [\bar{F}(x+y)]^{\gamma_{(s,n,\tilde{m}_n,k)}-1} \psi_{s-r}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right) f(x+y) \\ \times [\bar{F}(y)]^{\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(s,n,\tilde{m}_n,k)}-1} g_{r-1}(F(y)) f(y) dy,\ x \ge 0, \quad (5)$$

where, according to (4) for $r-1$,

$$\psi_{s-r}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right) = \int_{\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}^1 \psi_{s-r-1}(u) u^{m_{s-1}} du,\ r+1 \le s \le n, \quad (6)$$

with $\psi_0(t)=1$ and, for $r=1$, we have $f_{D_{(1,s,n,\tilde{m}_n,k)}}(x) = f_{X_{(s,n,\tilde{m}_n,k)}}(x)$.

Definition 2.1 ([12]). Let $\mathscr{X}$ and $\mathscr{Y}$ be subsets of $\mathbb{R}$. A function $\Lambda : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ is said to be totally positive of order 2 (TP_2) (reverse regular of order 2 (RR_2)) if $\Lambda(x_1,y_1)\Lambda(x_2,y_2) - \Lambda(x_1,y_2)\Lambda(x_2,y_1) \ge (\le) 0$, for $x_1 \le x_2$ in $\mathscr{X}$ and $y_1 \le y_2$ in $\mathscr{Y}$.

Lemma 2.2 ([13]). *Assume that Θ is a subset of the real line $\mathbb{R}$, and let U be a nonnegative random variable having a cdf belonging to the family $\mathscr{P} = \{\Xi(\cdot|\theta), \theta \in \Theta\}$ which satisfies that, for $\theta_1, \theta_2 \in \Theta$, $\Xi(\cdot|\theta_1) \leq_{st} (\geq_{st})\Xi(\cdot|\theta_2)$, whenever $\theta_1 \leq \theta_2$. Let $\phi(u,\theta)$ be a real valued function defined on $\mathbb{R} \times \Theta$, which is measurable in u for each θ such that $E_\theta[\phi(U,\theta)]$ exists. Then, $E_\theta[\phi(U,\theta)]$ is (i) increasing in θ, if $\phi(u,\theta)$ is increasing in θ and increasing (decreasing) in u; (ii) decreasing in θ, if $\phi(u,\theta)$ is decreasing in θ and decreasing (increasing) in u.*

The following lemmas play an important role for obtaining our main results.

Lemma 2.3. *For $s \geq r+1$, the function $\psi_{s-r}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)$ in (3) is TP_2 (RR_2) in (x,y) provided that any one of the conditions is satisfied: (a) f is logconvex (logconcave) and $m_i \geq 0$, (b) h is logconvex (logconcave) and $-1 \leq m_i < 0$.*

Proof. From (3), for any $y_1 \leq y_2$, we have

$$\frac{\psi_{s-r}\left(\frac{\bar{F}(x+y_2)}{\bar{F}(y_2)}\right)}{\psi_{s-r}\left(\frac{\bar{F}(x+y_1)}{\bar{F}(y_1)}\right)} = \frac{\int_{\mathbb{R}} I_{\{0\leq u\leq x\}}\psi_{s-r-1}\left(\frac{\bar{F}(u+y_2)}{\bar{F}(y_2)}\right)[\frac{\bar{F}(u+y_2)}{\bar{F}(y_2)}]^{m_{s-1}}\frac{f(u+y_2)}{\bar{F}(y_2)}du}{\int_{\mathbb{R}} I_{\{0\leq u\leq x\}}\psi_{s-r-1}\left(\frac{\bar{F}(u+y_1)}{\bar{F}(y_1)}\right)[\frac{\bar{F}(u+y_1)}{\bar{F}(y_1)}]^{m_{s-1}}\frac{f(u+y_1)}{\bar{F}(y_1)}du} = E_x[\phi(U,x)],$$

where I_A is the indicator function,

$$\phi(u,x) \propto \frac{\psi_{s-r-1}\left(\frac{\bar{F}(u+y_2)}{\bar{F}(y_2)}\right)}{\psi_{s-r-1}\left(\frac{\bar{F}(u+y_1)}{\bar{F}(y_1)}\right)}\left[\frac{\bar{F}(u+y_2)}{\bar{F}(u+y_1)}\right]^{m_{s-1}}\frac{f(u+y_2)}{f(u+y_1)} \tag{7}$$

$$= \frac{\psi_{s-r-1}\left(\frac{\bar{F}(u+y_2)}{\bar{F}(y_2)}\right)}{\psi_{s-r-1}\left(\frac{\bar{F}(u+y_1)}{\bar{F}(y_1)}\right)}\left[\frac{\bar{F}(u+y_2)}{\bar{F}(u+y_1)}\right]^{m_{s-1}+1}\frac{h(u+y_2)}{h(u+y_1)}, \tag{8}$$

and U is a nonnegative random variable having a cdf belonging to the family $\mathscr{P} = \{\Xi(\cdot|x,y_1), x, y_1 \in \mathbb{R}_+\}$ with corresponding pdf

$$\xi(u|x,y_1) = c(x,y_1)I_{\{0\leq u\leq x\}}\psi_{s-r-1}\left(\frac{\bar{F}(u+y_1)}{\bar{F}(y_1)}\right)\left[\frac{\bar{F}(u+y_1)}{\bar{F}(y_1)}\right]^{m_{s-1}}\frac{f(u+y_1)}{\bar{F}(y_1)}, \tag{9}$$

in which $c(x,y_1)$ is the normalizing constant. First, note that, for $x_1 \leq x_2$, $\frac{\xi(u|x_2,y_1)}{\xi(u|x_1,y_1)} \propto \frac{I_{\{0\leq u\leq x_2\}}}{I_{\{0\leq u\leq x_1\}}}$, is increasing in u because $I_{\{0\leq u\leq x\}}$ is TP_2 in (x,u). Let

$s-r=1$. According to (7), one can see that $\phi(u,x)$ is increasing (decreasing) in u when f is logconvex (logconcave) and $m_i \geq 0$. According to (8), $\phi(u,x)$ is increasing (decreasing) in u when f and h are logconvex (logconcave) and $-1 \leq m_i < 0$ (note that if h is logconcave then f is so). Also, $\phi(u,x)$ is constant with respect to x. Thus, the desired result follows by induction and Lemma 2.2. □

The proof of all the following new lemmas are similar to that of Lemma 2.3 and thus omitted.

Lemma 2.4. *Let $s-r \geq 1$ and Y be a nonnegative random variable having a cdf belonging to the family $\mathscr{P} = \{G(\cdot|x), x \in \mathbb{R}_+\}$ with corresponding pdf*

$$\begin{aligned}\zeta(y|x) &= c(x)[\bar{F}(x+y)]^{\gamma_{(s,n,\widetilde{m}_n,k)}-1}\psi_{s-r}\Big(\frac{\bar{F}(x+y)}{\bar{F}(y)}\Big)f(x+y) \\ &\quad \times[\bar{F}(y)]^{\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(s,n,\widetilde{m}_n,k)}-1}g_{r-1}(F(y))f(y), \qquad (10)\end{aligned}$$

where $c(x)$ is the normalizing constant. Then, $G(\cdot|x_1) \leq_{st} (\geq_{st}) G(\cdot|x_2)$, whenever $0 \leq x_1 \leq x_2$, provided that any one of the following conditions is satisfied:
i. f is logconvex (logconcave), $\gamma_{(s,n,\widetilde{m}_n,k)} \geq 1$, and $m_i \geq 0$;
ii. h is logconvex (logconcave) and $-1 \leq m_i < 0$.

Lemma 2.5. *For $s \geq 2$, the function $\psi_{s-1}(\frac{\bar{F}(x+y)}{\bar{F}(y)})/g_s(F(x))$ is increasing (decreasing) in x provided that any one of the following conditions is satisfied:*
i. f is logconvex (logconcave) and $m_i \geq 0$,
ii. h is logconvex (logconcave) and $-1 \leq m_i < 0$;

We also need some results for different parameters m_i and m'_i. From now on, we consider $\widetilde{m}'_{n'} = (m'_1, \ldots, m'_{n'-1})$ with $\gamma_{(r',n',\widetilde{m}'_{n'},k')} = k'+n'-r'+\sum_{j=r'}^{n'-1} m'_j > 0$.

Lemma 2.6. *Let $r \leq r'$. If $m'_{r'-i} \leq m_{r-i}$ for $1 \leq i \leq r-1$, then $\breve{g}_{r'}(t)/g_r(t)$ is increasing in t where g and $\breve{g}$ are defined in (2) with parameters m_i and m'_i, respectively. If $m'_{r-i} \geq m_{r-i}$ for $1 \leq i \leq r-1$, then $\breve{g}_r(t)/g_r(t)$ is decreasing in t.*

Lemma 2.7. *Suppose that ψ and $\check{\psi}$ are defined in (3) with parameters m_i and m'_i, respectively. Let $s'-r'=s-r$ and $s\leq s'$. For $s-r\geq 1$, the function*

$$\Delta_{(r,r',s,s')}(x,y)=\frac{\check{\psi}_{s'-r'}(\frac{\bar{F}(x+y)}{\bar{F}(y)})}{\psi_{s-r}(\frac{\bar{F}(x+y)}{\bar{F}(y)})}\cdot[\bar{F}(y)]^{m'_{s'-1}-m_{s-1}} \tag{11}$$

i. is increasing (decreasing) in y provided that $m'_j\leq m_i$ ($m'_j\geq m_i$) for any $i\leq j$, and any one of the following conditions is satisfied:

(a) f is logconvex and $m_i\geq 0$, (b) h is logconvex and $-1\leq m_i<0$;

ii. is increasing (decreasing) in x provided that $m'_j\leq m_i$ ($m'_j\geq m_i$) for any $i\leq j$.

3 Likelihood ratio ordering

In this section, we study the preservation of likelihood ratio ordering among spacings of GOS. It is worth mentioning that the direct studying of likelihood ratio ordering of spacings of GOS by means of its marginal pdf is rather complicated (since the pdf has not a closed form). Thus, some authors imposed some restrictions on the model. However, we obtain our main results directly. This enable us to have a more exible choice of parameters to compare the submodels of GOS.

Theorem 3.1. *Let $X_{(r,n,\widetilde{m}_n,k)}$, $r=1,\ldots,n$, $X_{(r',n',\widetilde{m}'_{n'},k')}$, $r'=1,\ldots,n'$, be GOS based on cdf F. If $r\leq r'$, $s\leq s'$ and $s'-r'=s-r$, then,*

$$D_{(r,s,n,\widetilde{m}_n,k)}\leq_{lr}D_{(r',s',n',\widetilde{m}'_{n'},k')}$$

provided that $m'_j\leq m_i$ for any $i\leq j$, $\gamma_{(s',n',\widetilde{m}'_{n'},k')}\leq\gamma_{(s,n,\widetilde{m}_n,k)}$, and any one of the following conditions is satisfied:

i. f is logconvex, $m_i\geq 0$, and

(a) $\gamma_{(s,n,\widetilde{m}_n,k)}\geq 1$, for $r\geq 2$, (b) $\gamma_{(s',n',\widetilde{m}'_{n'},k')}\geq 1$, for $r=1$;

ii. h is logconvex and $-1\leq m_i<0$.

Proof. We give the proof in two cases. *Case 1*: $r \geq 2$. From (5), we have

$$\frac{f_{D_{(r',s',n',\widetilde{m}'_{n'},k')}}(x)}{f_{D_{(r,s,n,\widetilde{m}_n,k)}}(x)} = E_x[\phi(Y,x)],$$

where

$$\begin{aligned}
\phi(y,x) &\propto [\bar{F}(x+y)]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-\gamma_{(s,n,\widetilde{m}_n,k)}} \frac{\breve{\psi}_{s'-r'}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{\psi_{s-r}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)} \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} \\
&\quad \times [\bar{F}(y)]^{(\gamma_{(r'-1,n',\widetilde{m}'_{n'},k')}-\gamma_{(s',n',\widetilde{m}'_{n'},k')})-(\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(s,n,\widetilde{m}_n,k)})}, \\
&= [\bar{F}(x+y)]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-\gamma_{(s,n,\widetilde{m}_n,k)}} \cdot \Delta_{(r,r',s,s')}(x,y) \cdot \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} \\
&\quad \times [\bar{F}(y)]^{(\gamma_{(r'-1,n',\widetilde{m}'_{n'},k')}-\gamma_{(s',n',\widetilde{m}'_{n'},k')})-(\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(s,n,\widetilde{m}_n,k)})-(m'_{s'-1}-m_{s-1})} \\
&= [\bar{F}(x+y)]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-\gamma_{(s,n,\widetilde{m}_n,k)}} \cdot \Delta_{(r,r',s,s')}(x,y) \cdot \frac{\breve{g}_{r'-1}(F(y))}{g_{r-1}(F(y))} \\
&\quad \times [\bar{F}(y)]^{(\Sigma_{j=r'-1}^{s'-2} m'_j)-(\Sigma_{j=r-1}^{s-2} m_j)}, \qquad (12)
\end{aligned}$$

$\Delta_{(r,r',s,s')}(x,y)$ defined in (11), and Y is a nonnegative random variable having a cdf belonging to the family $\mathscr{P} = \{G(\cdot|x), x \in \mathbb{R}_+\}$ with the pdf defined in (10). It is seen that the following hold in (12): The first term is increasing in x and y because of $\gamma_{(s',n',\widetilde{m}'_{n'},k')} \leq \gamma_{(s,n,\widetilde{m}_n,k)}$; The second term is increasing in x and y because of the conditions (i, ii) in the theorem, $m'_j \leq m_i$ for any $i \leq j$, and Lemma 2.7; The third term is increasing in y because of $r \leq r'$, $m'_j \leq m_i$ for any $i \leq j$, and Lemma 2.6; The fourth term is increasing in y because of $m'_j \leq m_i$ for any $i \leq j$. Further, according to the conditions (ia, ii) in the theorem and Lemma 2.4, we have $G(\cdot|x_1) \leq_{st} G(\cdot|x_2)$ for $x_1 \leq x_2$. Now, part (i) of Lemma 2.2 implies that $E_x[\phi(Y,x)]$ is increasing in x. *Case 2*: $r = 1$. From (1) and (5), we have

$$\begin{aligned}
\frac{f_{D_{(r',s',n',\widetilde{m}'_{n'},k')}}(x)}{f_{D_{(1,s,n,\widetilde{m}_n,k)}}(x)} &= \int_0^{+\infty} \frac{[\bar{F}(x+y)]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-1}}{[\bar{F}(x)]^{\gamma_{(s,n,\widetilde{m}_n,k)}-1}} \frac{\breve{\psi}_{s'-r'}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{g_s(F(x))} \frac{f(x+y)}{f(x)} \cdot \nu(y)dy \\
&= \int_0^{+\infty} [\frac{\bar{F}(x+y)}{\bar{F}(x)}]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-1} [\bar{F}(x)]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-\gamma_{(s,n,\widetilde{m}_n,k)}}
\end{aligned}$$

$$\times \frac{\breve{\psi}_{s'-r'}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{g_s(F(x))} \frac{f(x+y)}{f(x)} \cdot \nu(y)dy \tag{13}$$

$$= \int_0^{+\infty} [\frac{\bar{F}(x+y)}{\bar{F}(x)}]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}} [\bar{F}(x)]^{\gamma_{(s',n',\widetilde{m}'_{n'},k')}-\gamma_{(s,n,\widetilde{m}_n,k)}}$$

$$\times \frac{\breve{\psi}_{s'-r'}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{g_s(F(x))} \frac{h(x+y)}{h(x)} \cdot \nu(y)dy, \tag{14}$$

where $\nu(y)$ does not depend on x. Now, according to the conditions of theorem, to prove that (13) and (14) is increasing in x, it is sufficient to show that $\breve{\psi}_{s'-r'}(\frac{\bar{F}(x+y)}{\bar{F}(y)})/g_s(F(x))$ is increasing in x. To do this, we consider it as

$$\frac{\breve{\psi}_{s'-r'}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{g_s(F(x))} = \frac{\breve{\psi}_{s'-r'}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{\psi_{s-r}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)} \frac{\psi_{s-r}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right)}{g_s(F(x))}. \tag{15}$$

So, the first and second term is increasing in x by Lemma 2.7(ii) and Lemma 2.5 with $r = 1$, respectively. Therefore, the proof is completed. □

Using Lemmas 2.2, 2.4, 2.6 and some calculations, we give the following.

Theorem 3.2. *Let* $X_{(r,n,\widetilde{m}_n,k)}$, $r = 1,\ldots,n$, $X_{(r',n',\widetilde{m}'_{n'},k')}$, $r' = 1,\ldots,n'$, *be GOS based on cdf* F. *If* $r \le r'$, *then,* $D_{(r,r,n,\widetilde{m}_n,k)} \ge_{lr} D_{(r',r',n',\widetilde{m}'_{n'},k')}$, *provided that, for* $r \ge 2$, $m'_j \le m_i$ *for any* $i \le j$, $\gamma_{(r',n',\widetilde{m}'_{n'},k')} = \gamma_{(r,n,\widetilde{m}_n,k)}$, *for* $r = 1$, $\gamma_{(r',n',\widetilde{m}'_{n'},k')} \ge \gamma_{(1,n,\widetilde{m}_n,k)}$, *and any one of the following conditions is satisfied:*
i. f *is logconcave,*

(a) $\gamma_{(r,n,\widetilde{m}_n,k)} \ge 1$ *and* $m_i \ge 0$, *for* $r \ge 2$, *(b)* $\gamma_{(r',n',\widetilde{m}'_{n'},k')} \ge 1$, *for* $r = 1$;

ii. h *is logconcave, and, for* $r \ge 2$, $-1 \le m_i < 0$.

Using Lemmas 2.2, 2.4, 2.5, 2.6 and some calculations, we give the following.

Theorem 3.3. *Let* $X_{(r,n,\widetilde{m}_n,k)}$, $r = 1,\ldots,n$, $X_{(r',n',\widetilde{m}_{n'},k')}$, $r' = 1,\ldots,n'$, *be GOS based on cdf* F. *If* $r+1 \le r'$ *and* $s'-r' = s-r$, *then,* $D_{(r,s,n,\widetilde{m}_n,k)} \ge_{lr} D_{(r',s',n',\widetilde{m}_{n'},k')}$, *provided that, for* $r \ge 2$, $m_j \le m_i$ *for any* $i \le j$, $\gamma_{(s',n',\widetilde{m}_{n'},k')} = \gamma_{(s,n,\widetilde{m}_n,k)}$, *for*

$r=1$, $\gamma_{(s',n',\widetilde{m}_{n'},k')} \geq \gamma_{(s,n,\widetilde{m}_n,k)}$, *and any one of the following conditions is satisfied:*

i. f *is logconcave,* $m_i \geq 0$, *and*

(a) $\gamma_{(s,n,\widetilde{m}_n,k)} \geq 1$, *for* $r \geq 2$, *(b)* $\gamma_{(s',n',\widetilde{m}_{n'},k')} \geq 1$, *for* $r=1$*;*

ii. h *is logconcave and* $-1 \leq m_i < 0$.

Using Lemmas 2.2, 2.4, 2.5, 2.6, 2.7 and some calculations, we give the following.

Theorem 3.4. *Let* $X_{(r,n,\widetilde{m}_n,k)}$, $r=1,\ldots,n$, $X_{(r',n',\widetilde{m}_{n'},k')}$, $r'=1,\ldots,n'$, *be GOS based on cdf* F. *If* $r' \leq r-1$ *and* $s'-r'=s-r$, *then,* $D_{(r,s,n,\widetilde{m}_n,k)} \leq_{lr} D_{(r',s',n',\widetilde{m}_{n'},k')}$, *provided that, for* $r \geq 3$, $m_j \geq m_i$ *for any* $i \leq j$, $\gamma_{(s',n',\widetilde{m}_{n'},k')} = \gamma_{(s,n,\widetilde{m}_n,k)}$, *for* $r=2$, $\gamma_{(s',n',\widetilde{m}_{n'},k')} \leq \gamma_{(s,n,\widetilde{m}_n,k)}$, *and any one of the following conditions is satisfied:*

i. f *is logconcave,* $\gamma_{(s,n,\widetilde{m}_n,k)} \geq 1$ *and* $m_i \geq 0$*;*

ii. h *is logconcave and* $-1 \leq m_i < 0$.

Remark 3.5. (1) [16] proved the statement of Theorem 1 (in their separate Theorems 3.1, 3.2, 3.3 and Corollary 3.4) under the conditions as follows:

$$k=k',\ n=n',\ m_i=m_i',\ m_i \text{ is decreasing in i, and } r'-r \geq n'-n. \qquad (16)$$

By choosing $s=r+p-1$ and $s'=r'+p-1$, one can see that (16) implies the conditions in Theorem 1;

(2) [8] proved the statement of Theorems 2 and 3.3 under very stronger conditions $m_1=\ldots=m_{n-1}$ and $m_i=m_i'$. Additional of these restrictions, by choosing $s=r+p-1$ and $s'=r'+(p+1)-1$ with $r'=r-1$ in our Theorem 4.4, one can see that Theorem 4.4 of [8] is a special case of Theorem 4.4.

4 Applications

Sequential $(n-k+1)$**-out-of-**n **system**. In this system, successive failure times of components are observed which are called sequential order statistics (SOS). The system collapses after the k-th failure so that the k-th SOS describes the system lifetime. After the failure of the i-th component, the distribution of the lifetimes of the remaining components in the system is adjusted by a parameter α_i (cf. [6]). This reflects both a damage caused by the previous failures and a higher load imposed on the remaining components leading possibly to shorter residual life. SOS under proportional hazard rates are included in GOS. Indeed, the specific choice of distribution functions $F_i(x) = 1-(1-F(x))^{\alpha_i}$, with a cdf F and positive real numbers $\alpha_1,\dots,\alpha_n$ leads to the model of GOS with parameters $k = \alpha_n$ and $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1$ (and hence $\gamma_i = (n-i+1)\alpha_i$). The flexible results of the paper enable us to compare spacings of two sequential systems with different parameters α_i and α_i' in likelihood ratio orders when the components have the same life time distribution.

Progressive Type-II censored order statistics. A progressively censored life test involves N items with i.i.d. lifetimes placed simultaneously on test. At the time of the i-th failure ($1 \le i \le n$), R_i surviving units are randomly withdrawn from the test. Progressively Type-II censored order statistics (PCOS) arising from such a reliability experiment correspond to GOS with parameters $m_i = R_i \in \mathbb{N}_0$, $i = 1,\dots,n-1$, and $k = R_n + 1$. The vector $\tilde{R} = (R_1,\dots,R_n)$ is called censoring plan (cf. [5]). Our results can be applied to compare spacings of PCOS in two tests with different censoring plans $\tilde{R}$ and $\tilde{R}'$ in likelihood ratio orders when n failures are observed and the components have the same lifetime distributions.

Finally, we note that the results of this paper can be applied for other submodels of GOS such as record values, Pfeifers record values and order statistics under multivariate imperfect repair.

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Mixture Representations of Lifetime in Coherent Systems with Dependent Components: Stochastic Comparisons

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Abstract: In this talk we consider lifetime of coherent systems as a generalized finite mixture model, which formed by dependent and identically distributed (d.i.d) components. We then establish some general results for the comparisons of two such generalized finite mixture models in two different cases: (i) when two mixture models are formed from two random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ and having same weights, (ii) when two mixture models are formed with the same random vectors and having different weights. Because the lifetimes of k-out-of-n systems and coherent systems are special cases of the considered mixture model, we established results and then used to compare the lifetimes k-out-of-n systems and of coherent systems with respect to various stochastic orderings.

Keywords: Coherent system, Copula function, Generalized mixture model, k-out-of-n system, Stochastic orders.

1 Introduction

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of non-negative d.i.d. random variables with an absolutely continuous distribution F, survival function $\overline{F} = 1 - F$, and density function f. The joint distribution function of $\mathbf{X}$ is given by

$$F_{\mathbf{X}}(\boldsymbol{x}) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = \mathbb{C}(F(x_1), \ldots, F(x_n)), \quad (1)$$

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where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbb{C}$ is the multivariate distribution copula on $[0,1]^n$ with uniformly distributed marginals on $[0,1]$. The joint survival (or reliability) function of $\boldsymbol{X}$ has the form

$$\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) = \mathbb{P}(X_1 > x_1, \ldots, X_n > x_n) = \widehat{\mathbb{C}}(\overline{F}(x_1), \ldots, \overline{F}(x_n)),$$

where $\overline{F}$ is the survival function; it is also referred to as a reliability copula (see Nelsen (2006)). Now, let $\overline{\mathbb{K}}_i(\overline{F}(x)) = \widehat{\mathbb{C}}(\overline{F}(x)\mathbf{1}_i, \mathbf{1}_{n-i})$ denote the survival function of the series system $X_{1:i} = \min(X_1, \ldots, X_i)$, where the entries of both $\mathbf{1}_i$ and $\mathbf{1}_{n-i}$ are all ones, with $\overline{\mathbb{K}}_1(\overline{F}(x)) = \overline{F}(x)$ and $\overline{\mathbb{K}}_n(\overline{F}(x)) = \widehat{\mathbb{C}}(\overline{F}(x), \ldots, \overline{F}(x))$.

We now define the survival function of a generalized finite mixture model (wherein the mixing proportions may be negative) from these i-dimensional marginals of $\widehat{\mathbb{C}}$ as follows:

$$\overline{H}_{\boldsymbol{X},\boldsymbol{a}}(\overline{F}(x)) = \sum_{i=1}^{n} a_i \overline{\mathbb{K}}_i(\overline{F}(x)), \tag{2}$$

where $\boldsymbol{a} = (a_1, \ldots, a_n)$ are some real numbers (weights) such that $\sum_{i=1}^{n} a_i = 1$. Note that if all the weights are positive, then the mixture model in (2) becomes a pure mixture model. If some of the weights are negative, then we have the mixture in (2) to be a generalized mixture model. Suppose $u = \overline{F}(x)$ for all $u \in [0,1]$; then, (2) can be rewritten as

$$\overline{H}_{\boldsymbol{a}}(u) = \sum_{i=1}^{n} a_i \overline{\mathbb{K}}_i(u),$$

where $\overline{H}_{\boldsymbol{a}}(u)$ is a proper distribution function from $[0,1]$ to $[0,1]$ with $H_{\boldsymbol{a}}(0) = 0$ and $H_{\boldsymbol{a}}(1) = 1$. Moreover, the distribution function corresponding to the generalized mixture model in (2) is given by

$$H_{\boldsymbol{X},\boldsymbol{a}}(\overline{F}(x)) = \sum_{i=1}^{n} a_i \mathbb{K}_i(\overline{F}(x)),$$

where $H_{\boldsymbol{X},\boldsymbol{a}}(\overline{F}(x)) = 1 - \overline{H}_{\boldsymbol{X},\boldsymbol{a}}(\overline{F}(x))$ and $\mathbb{K}_i(\overline{F}(x)) = 1 - \overline{\mathbb{K}}_i(\overline{F}(x))$.

We specificity establish some general results for the comparison of two generalized mixture models in (2) in the sense of hazard rate and reversed hazard rate orders.

We first briefly review some notions of stochastic orderings and ageing properties that are used in the subsequent sections of this paper. Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}^+ = [0, +\infty)$. Throughout this paper, we use increasing to mean non-decreasing and decreasing to mean non-increasing.

Definition 1.1. Let X and Y be two non-negative random variables with density functions f and g, distribution functions F and G, survival functions $\overline{F} = 1-F$ and $\overline{G} = 1-G$, hazard rate functions $r_X = \frac{f}{\overline{F}}$ and $r_Y = \frac{g}{\overline{G}}$, and reversed hazard rate functions $\tilde{r}_X = \frac{f}{F}$ and $\tilde{r}_Y = \frac{g}{G}$, respectively. Then:

- X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $\frac{g(x)}{f(x)}$ is increasing in x;
- X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\frac{\overline{G}(x)}{\overline{F}(x)}$ is increasing in x, or, equivalently, $r_X(x) \geq r_Y(x)$ for all x;
- X is said to be smaller than Y in the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $\frac{G(x)}{F(x)}$ is increasing in x, or, equivalently, $\tilde{r}_X(x) \leq \tilde{r}_Y(x)$ for all x;
- X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all $x \in \mathbb{R}^+$, or equivalently, $\mathsf{E}[\phi(X)] \leq [\geq]\mathsf{E}[\phi(Y)]$ for any increasing [decreasing] function $\phi : \mathbb{R} \to \mathbb{R}$ for which the involved expectations exist.

It is well-known that

$$X \leq_{lr} Y \Longrightarrow X \leq_{hr[rh]} Y \Longrightarrow X \leq_{st} Y, \tag{3}$$

but neither reversed hazard rate order nor hazard rate order implies the other. For more detailed discussions on the above stochastic orderings, one may refer to the books by [11] and [9].

Several authors have studied stochastic comparisons of mixture models; see [1], [5], [2], [4], [3], [6], [7] and [8]. Recently, Hernandez (2007) and [10] obtained.

In this work, we first consider two statistical models $H_{\boldsymbol{X},\boldsymbol{a}}$ and $H_{\boldsymbol{Y},\boldsymbol{a}}$ having different components and same weights and establish some ordering results between them in the sense of hazard rate and reversed hazard rate orders.

Next, we consider $H_{\boldsymbol{X},\boldsymbol{a}}$ and $H_{\boldsymbol{X},\boldsymbol{b}}$ having the same components with different weights and establish results with respect to different stochastic orders.

2 Main results

In this section, we obtain some general results for the comparison of general mixture models in (2) in the following two cases: two mixture models formed from two sets of random vectors, $\boldsymbol{X}$ and $\boldsymbol{Y}$, with the same weights, and two mixture models formed from one random vector of components, $\boldsymbol{X}$, with different weights.

2.1 Stochastic comparisons of mixture models with two different random vectors

Theorem 2.1. *Let $H_{\boldsymbol{X},\boldsymbol{a}}$ and $H_{\boldsymbol{Y},\boldsymbol{a}}$ be two generalized finite mixture models with d.i.d. components $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, and having the same copula function. If*

(i) $\frac{u\overline{H}'_{\boldsymbol{a}}(u)}{\overline{H}_{\boldsymbol{a}}(u)}$ *is decreasing in u for all $u \in (0,1)$, and*

(ii) $X_1 \leq_{hr} Y_1$,

then $H_{\boldsymbol{X},\boldsymbol{a}} \leq_{hr} H_{\boldsymbol{Y},\boldsymbol{a}}$.

Theorem 2.2. *Let $H_{\boldsymbol{X},\boldsymbol{a}}$ and $H_{\boldsymbol{Y},\boldsymbol{a}}$ be two generalized finite mixture models with d.i.d. components $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, and having the same copula function. If*

(i) $\frac{(1-u)\overline{H}'_{\boldsymbol{a}}(u)}{1-\overline{H}_{\boldsymbol{a}}(u)}$ *is increasing in u for all $u \in (0,1)$, and*

(ii) $X_1 \leq_{rh} Y_1$,

then $H_{\boldsymbol{X},\boldsymbol{a}} \leq_{rh} H_{\boldsymbol{Y},\boldsymbol{a}}$.

2.2 Stochastic comparisons of mixture models with the same random vector

We now consider two mixture models, $H_{\boldsymbol{X},\boldsymbol{a}}$ and $H_{\boldsymbol{X},\boldsymbol{b}}$, having the same d.i.d. components but with different vectors of weights, $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. We then establish some sufficient conditions for the comparison of the two mixtures models with respect to hazard rate and reversed hazard rate orderings.

Theorem 2.3. *Let $H_{\boldsymbol{X},\mathbf{a}}$ and $H_{\boldsymbol{X},\mathbf{b}}$ be two generalized finite mixture models with d.i.d. components $\boldsymbol{X}$ and vectors of weights $\mathbf{a}$ and $\mathbf{b}$, respectively. If*

(i) $\frac{u\overline{\mathbb{K}}'_j(u)}{\overline{\mathbb{K}}_j(u)}$ *is increasing in j for all $1 \leq j \leq n$, and*

(ii) $a_i b_j \leq a_j b_i$ *for all $1 \leq i \leq j \leq n$.*

then $H_{\boldsymbol{X},\mathbf{a}} \leq_{hr} H_{\boldsymbol{X},\mathbf{b}}$,

Theorem 2.4. *Let $H_{\boldsymbol{X},\mathbf{a}}$ and $H_{\boldsymbol{X},\mathbf{b}}$ be two generalized finite mixture models with d.i.d. components $\boldsymbol{X}$ and vectors of weights $\mathbf{a}$ and $\mathbf{b}$, respectively. If*

(i) $\frac{(1-u)\mathbb{K}'_j(u)}{1-\overline{\mathbb{K}}_j(u)}$ *is increasing in j for all $1 \leq j \leq n$, and*

(ii) $a_i b_j \leq a_j b_i$ *for all $1 \leq i \leq j \leq n$.*

then $H_{\boldsymbol{X},\mathbf{a}} \leq_{rh} H_{\boldsymbol{X},\mathbf{b}}$,

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Maintenance of Continuously Monitored Multi-unit Systems

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Abstract: The main purpose of this work is to model a continuously monitored deteriorating system by using a bivariate Birth/Birth-Death process. The system consists of M identical, independent units, each subject to gradual deterioration. The production rate of each unit varies in different working states and the demand rate of the system is assumed to be constant. The cost of sending maintenance crew to perform maintenance is high, so simultaneous replacement of several units can be cost-effective. Maintenance is initiated when the process makes a transition into a state with r failed units. The maintenance policy prescribes corrective replacement of failed units as well as preventive maintenance of unhealthy units. The optimal maintenance policy is derived such that the long-run expected average cost per unit time is minimized, according to the renewal theory. Finally, a practical example of a multi-unit system is provided.

Keywords: Condition-based maintenance, Continuously monitoring, Multi-unit systems.

1 Introduction

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Production systems are getting much more complicated today, and aging is becoming a significant issue. Traditional approaches of system reliability and maintenance which have primarily been performed on a corrective basis, independently of the deterioration progression, are not adequate to address these developments. Corrective maintenance will only be carried out after a failure occurs, which may result in substantial production losses. The idea of maintenance has grown to the point where it is now used to prevent failure and maintain the system in proper operation condition ([3]). Generally, preventive maintenance policies are classified into two categories: age-based and condition-based maintenance (CBM). The literature on age-based maintenance policies is widespread; see for example [4] and [10].

In comparison with the age-based maintenance, CBM is mainly taken into account as a significant maintenance strategy that collects and analyses real-time information about the system state and suggests maintenance actions based on the system's current condition. The existing CBM models for a single unit system were reviewed in [2]. Maintenance issues in multi-unit systems are much more complex than maintenance policies in single-unit systems. Such systems are more applicable to real-world scenarios, but research in this area is limited. A CBM model for multi-unit systems subject to stochastic deteriorations was studied by [8]. They presented a preventive opportunistic maintenance policy for a two-unit system considering economic dependency between the units. [5] proposed an optimization model which is defined by a combination of CBM and age information. Preventive maintenance is performed on the two units, one of which is the system's key component and is subject to condition monitoring, and only the age information for the second unit is available.

Maintenance activities affect production, so it is critical to integrate maintenance and production. A breakdown of machine operation causes production to be disrupted, downtime, and missed production costs rise [1]. Several research studies have focused on production system maintenance policies. [7]

considered the joint optimization of the CBM and inventory decisions for a multi-component system with a shared pool of spares. A different CBM policy was developed by [9] for a deteriorating multi-unit system modeled based on three states. The authors considered a system production level threshold for initiating preventive maintenance using discrete-time monitoring.

To our knowledge, no papers have been published considering a bivariate homogeneous Markov process for a multi-unit system maintenance modeling. In this work, we develop the bivariate Birth/Birth-Death process, introduced by [6], for modeling the number of units in an unhealthy state and a failure state. Considering that, we obtain the optimal maintenance level under different conditions and study the effect of the lost demand cost rate on the optimal policy. The remainder of the paper is organized as follows. The details of the proposed model are summarized in 2. A computational algorithm to obtain maintenance level is developed in 3. Finally, an illustrative example is presented in 4.

2 Model Formulation

We investigate the model formulation under the following assumptions.

Assumptions

- A system is composed of a large number, M, of identical units which are operating in parallel and subject to deteriorate.
- Unit's state can be categorized into one of the three states: a healthy state (state 0), unhealthy state (state 1), and a failure state (state 2) where all states are observable for each unit.
- Units can make transition from $0 \longrightarrow 1$ with probability p_{01}, and then from $1 \longrightarrow 2$ with probability p_{12}, or from $0 \longrightarrow 2$ with probability p_{02}, where $p_{01} + p_{02} = 1$ and $p_{12} = 1$.

- The sojourn time in state i, $i = \{0, 1\}$, has an exponential distribution with an unknown parameter λ_i.

2.1 Birth/Birth-Death Process

Let $N_0(t)$, $N_1(t)$, and $N_2(t)$ be the number of units in state 0, the number of units in state 1, and the number of units in state 2 at time t, respectively. Because $N_0(t) + N_1(t) + N_2(t)$ remains constant, we only consider a bivariate homogeneous Markov chain $\{(N_2(t), N_1(t)), t \geq 0\}$ with the state space $S = \{(i, j) : 0 \leq i, j \leq M, 0 \leq i + j \leq M\}$, to trace $N_2(t)$ and $N_1(t)$. In other words, if $N_2(t) = i$ and $N_1(t) = j$, the number of units in state 0 will be $M - i - j$ at time t. For this purpose, we are going to use a subclass of competition processes with two interacting populations of operating units, $(N_2(t), N_1(t))$, called birth/birth-death process whose first population is increasing $(N_2(t))$. All possible transitions occur with the following probabilities:

$$P((N_2(t+dt), N_1(t+dt)) = (i, j+1) | (N_2(t), N_1(t)) = (i, j)) \\ = \lambda_{ij}^{(1)} dt + o(dt),$$

$$P((N_2(t+dt), N_1(t+dt)) = (i+1, j) | (N_2(t), N_1(t)) = (i, j)) \\ = \lambda_{ij}^{(2)} dt + o(dt),$$

$$P((N_2(t+dt), N_1(t+dt)) = (i+1, j-1) | (N_2(t), N_1(t)) = (i, j)) \\ = \gamma_{ij} dt + o(dt),$$

$$P((N_2(t+dt), N_1(t+dt)) = (i, j) | (N_2(t), N_1(t)) = (i, j)) = \\ 1 - (\lambda_{ij}^{(1)} + \lambda_{ij}^{(2)} + \gamma_{ij}) dt + o(dt),$$

where $\lambda_{ij}^{(1)} = (M - i - j) p_{01} \lambda_0$ is the transition rate from state 0 to state 1, $\lambda_{ij}^{(2)} = (M - i - j) p_{02} \lambda_0$ is the transition rate from state 0 to state 2, and $\gamma_{ij} = j p_{12} \lambda_1$ is the failure rate of units in state 1, given i units in state 2 and j units in state 1. The leaving rates of the states can be obtained by:

$$\nu_{ij} = \begin{cases} \lambda_{ij}^{(1)} + \lambda_{ij}^{(2)} + \gamma_{ij} = (M-i-j)\lambda_0 + j\lambda_1, & 0 \le i \le M-1,\ 1 \le j \le M-i-1, \\ \lambda_{ij}^{(1)} + \lambda_{ij}^{(2)} = (M-i)\lambda_0, & 0 \le i \le M-1,\ j=0, \\ \gamma_{ij} = j\lambda_1, & 0 \le i \le M-1,\ j = M-i, \\ 0, & i=M,\ j=0. \end{cases} \tag{1}$$

The state of the continuous-time Markov chain $\{(N_2(t), N_1(t))\}$ just after a state transition is described by the discrete-time Markov chain $\{(N_{2,n}, N_{1,n})\}$ whose one-step transition probabilities $p_{(i,j),(i',j')}$ at the end of sojourn times are derived as follows:

- When $0 \le i \le M-1,\ 1 \le j \le M-i-1$, the one-step transition probabilities are given by:

$$p_{(i,j),(i',j')} = \begin{cases} \dfrac{\lambda_{ij}^{(1)}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}+\gamma_{ij}}, & i'=i,\ j'=j+1 \\ \dfrac{\lambda_{ij}^{(2)}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}+\gamma_{ij}}, & i'=i+1,\ j'=j \\ \dfrac{\gamma_{ij}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}+\gamma_{ij}}, & i'=i+1,\ j'=j-1 \end{cases} \tag{2}$$

- When $0 \le i \le M-1,\ j=0$, the one-step transition probabilities are given by:

$$p_{(i,j),(i',j')} = \begin{cases} \dfrac{\lambda_{ij}^{(1)}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}}, & i'=i,\ j'=j+1 \\ \dfrac{\lambda_{ij}^{(2)}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}}, & i'=i+1,\ j'=j \end{cases} \tag{3}$$

- When $0 \le i \le M-1,\ j=M-i$, a transition is possible to the state with $i'=i+1,\ j'=j-1$ with probability 1:

$$p_{(i,j),(i',j')} = 1 \tag{4}$$

Uniformization method can obtain the probabilities $P_{(i,j),(i',j')}(t)$ via:

$$P_{(i,j),(i',j')}(t) = \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \bar{p}^{(n)}_{(i,j),(i',j')}, \tag{5}$$

where the probabilities $\bar{p}^{(n)}_{(i,j),(i',j')}$ can be recursively computed from

$$\bar{p}^{(n)}_{(i,j),(i',j')} = \sum_{(k,h)\in S} \bar{p}^{(n-1)}_{(i,j),(k,h)} \bar{p}_{(k,h),(i',j')}, \qquad n = 1,2,... \tag{6}$$

starting with $\bar{p}^{(0)}_{(i,j),(i,j)} = 1$ and $\bar{p}^{(0)}_{(i,j),(i',j')} = 0$ for $(i,j) \neq (i',j')$.

3 Maintenance Model

In this section, we investigate the maintenance and replacement policy under the following assumptions:

- States of the degrading system can be described by the bivariate birth/birth-death process, $\{(N_2(t),N_1(t)),t \geq 0\}$, having the initial state $(N_2(0),N_1(0)) = (0,0)$.
- Monitoring of the system is continuous and perfect, i.e., it reveals instantaneously the true state of the system.
- Maintenance is initiated when the process $(N_2(t),N_1(t))$ makes a transition into a state with r failed units while units in states 0 and 1 are working during the lead time interval $[0,u]$.
- Maintenance policy suggests corrective replacement of the failed units and preventive replacement of the units in state 1. After maintenance, all the units are in state 0.
- Replacements are assumed to be instantaneous and perfect which bring the units to the as-good-as-new state.
- The optimal policy minimizes the long-run expected average cost per unit time.

We also consider the following cost components:

- C_0, C_1: Operating cost rates of each unit in state 0 and state 1, respectively
- C_F: Failure replacement cost of each unit
- C_P: Preventive replacement cost of each unit
- C_D: Cost rate of loss production when production rate of the system is below the demand rate
- C_E: Profit rate from excess production
- C_K: Cost of sending maintenance team to do maintenance

3.1 Development of the Optimal Policy

The objective here is to derive an optimal replacement policy such that the long-run expected average cost per unit time is minimized. According to the renewal theory, it is given by:

$$g = \frac{\text{Expected Cycle Cost}}{\text{Expected Cycle Length}} = \frac{E(CC)}{E(CL)}. \tag{7}$$

Let E_{ij} be the time instant when the system is identified to be in state $(i, j) \in S$ and $X_\delta(i, j)$ be the expected time from E_{ij} to the completion of replacement under policy δ. It can be calculated recursively using the following equations: when $0 \leq i < r,\ j = 0$,

$$X_\delta(i,j) = \frac{1}{\lambda_{ij}^{(1)} + \lambda_{ij}^{(2)}} + \frac{\lambda_{ij}^{(1)}}{\lambda_{ij}^{(1)} + \lambda_{ij}^{(2)}} X_\delta(i, j+1) + \frac{\lambda_{ij}^{(2)}}{\lambda_{ij}^{(1)} + \lambda_{ij}^{(2)}} X_\delta(i+1, j),$$

when $0 \leq i < r,\ j = M - i$,

$$X_\delta(i,j) = \frac{1}{\gamma_{ij}} + X_\delta(i+1, j-1),$$

when $0 \leq i < r,\ \ 1 \leq j \leq M - i - 1$,

$$X_\delta(i,j) = \frac{1}{\lambda_{ij}^{(1)} + \lambda_{ij}^{(2)} + \gamma_{ij}} + \frac{\lambda_{ij}^{(1)}}{\lambda_{ij}^{(1)} + \lambda_{ij}^{(2)} + \gamma_{ij}} X_\delta(i, j+1)$$

$$+\frac{\lambda_{ij}^{(2)}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}+\gamma_{ij}}X_\delta(i+1,j)+\frac{\gamma_{ij}}{\lambda_{ij}^{(1)}+\lambda_{ij}^{(2)}+\gamma_{ij}}X_\delta(i+1,j-1),$$

Let $Y_\delta(i,j)$ be the expected cost from E_{ij} to the completion of replacement under policy δ and $C_{(i,j)}$ be a constant cost rate associated with each state (i,j) of the process $(N_1(t),N_2(t))$, given by:

$$C_{(i,j)} = (M-i-j)\times C_0 + j\times C_1 + C_D\times \max\{0, D-[(M-i-j)\times p_0 + j\times p_1]\}$$
$$-C_E\times \max\{0,[(M-i-j)\times p_0 + j\times p_1]-D\}.$$

$Y_\delta(i,j)$ can also be obtained recursively using the idea of the above equations. Now, we define a cycle as a period of time from starting time until the end of the replacement time. Then, from Eq. 7, the long-run expected average cost per unit time for policy δ is equal to:

$$g=\frac{E(CC)}{E(CL)}=\frac{Y_\delta(0,0)}{X_\delta(0,0)}. \quad (8)$$

We want to find δ^* such that:

$$g^*=\inf_{\delta\in\Delta}\frac{Y_\delta(0,0)}{X_\delta(0,0)}=\frac{Y_{\delta^*}(0,0)}{X_{\delta^*}(0,0)}, \quad (9)$$

where Δ is the set of all policies δ.

4 Numerical Example

Numerical examples to illustrate the entire optimization procedure are presented in this section where the selected parameter values are given by [9]. We consider a wind farm consisting of $M=30$ wind turbines subject to on-line monitoring of their gearboxes. We assume that the deterioration process of each gearbox follows a continuous time Markov chain with two operating states $\{0,1\}$ and a failure state $\{2\}$ which is absorbing. The sojourn time in states $\{0\}$ and $\{1\}$ have an exponential distribution with parameters $\lambda_0=0.4345\times10^{-3}$ and $\lambda_1=0.3036\times10^{-3}$, respectively and the transition probability p_{01} equals to 0.60.

Table 1: Optimal Decision Variable and Average Cost for $M = 30$

C_D \ C_E		0	0.02	0.04
0.4	r^*	8	3	2
	Average cost	455.42	401.39	328.49
0.6	r^*	7	3	2
	Average cost	461.52	402.49	329.30
0.8	r^*	6	3	2
	Average cost	465.66	403.59	330.10
1	r^*	5	3	2
	Average cost	468.97	404.69	330.91

To optimize maintenance strategies, we study a system with the above considerations and also assume that each turbine is working in the wind farm with a rated capacity of 1.5 MW and capacity factors of 30% and 22% in state 0 and 1, respectively. These capacity factors result in the output of $p_0 = 450$ kWh for the units in state 0, and the output of $p_1 = 330$ kWh for the units in state 1, respectively. We assume the demand rate of the wind farm is $D = 9000$ kWh, for systems with $M = 30$ number of units, respectively.

Failure cost is $C_F = \$78368$ *andpreventivemaintenancecostis*C˙p=$8182 . To analyze the sensitivity of the optimal policies for different values of the lost production cost rate C_D and profit rate from the excess production C_E, they are taken as follows: $C_D = 0.4,\ 0.6,\ 0.8,\ 1$ and $C_E = 0,\ 0.02,\ 0.04$. Another cost parameter is C_K which is considered to be 3000 per visit of the farm with $M = 30$ units, respectively. We assume that the lead time is $u = 100$, which is constant.

The results of optimal decision variable and average cost for various cost rate values and number of units are presented in Table 1. According to this table, we can observe that for a give C_D, the threshold to perform maintenance decreases by the increase of C_E and for a given C_E, when the lost demand cost rate increases, the threshold to perform maintenance decreases or remains constant.

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Estimation of $P(Y < X)$ for the Skew-normal Distribution Under Progressive Type-II Censored Samples

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Abstract: This paper deals the inference of $R = P(Y < X)$ for skew-normal distribution based on progressively Type-II censored samples. To do this, the MLE of R and the Bayes estimator and the corresponding credible interval are extracted using Gibbs sampling. The MLE, the Bayes estimator and confidence interval are acquired in the skew-normal distribution with known scale parameter explicitly. Finally, the inference of the reliability parameter using the proposed methods is performed on artificial and real datasets.

Keywords: Bayesian and classical inference, Progressive Type-II censoring, Stress-strength.

1 Introduction

The reliability parameter, $R = P[Y < X]$, is one of the important problems in reliability theory. The strength, X, and the stress, Y, are considered as random variables. The system fails whenever X is less than Y.
Several authors have studied the inference of the R by using different classes

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of statistical distribution. Kotz et al. [1] provided a comprehensive review of the development of these studies until 2003.

The Type-II progressive censoring scheme, one of the most widespread censorship schemes used in reliability, is as follows: Let n units are subjected to a test and set $m \leq n$ number of failures using a given scheme $(R_1, R_2, \ldots, R_m)$. Randomly, R_1 of the $n-1$ surviving units are chosen at the time of the first failure and removed from the experiment. Likewise, R_2 of the $n-R_1-2$ surviving units are withdrawn randomly at the time of the second failure and the rest. Finally, at the time of the mth failure, all remaining surviving observations, $n-R_1-R_2-\ldots-R_{m-1}-m$ are eliminated. For more information about progressively censoring, see [2].

Cumulative distribution function (CDF) of skew-normal (SN) distribution is as follows

$$F(x;\alpha,\sigma) = 1-[1-\Phi(\frac{x}{\sigma})]^{\alpha}, \quad -\infty < x < +\infty, \tag{1}$$

where $\sigma, \alpha \in (0,+\infty)$ and $\Phi(.)$ denote the CDF of standard normal distribution. The probability density function (PDF) related to the CDF (1) is

$$f(x;\alpha,\sigma) = \frac{\alpha}{\sigma}\phi(\frac{x}{\sigma})[1-\Phi(\frac{x}{\sigma})]^{\alpha-1}, \quad -\infty < x < +\infty, \tag{2}$$

where $\phi(.)$ is the PDF of standard normal distribution. α and σ are the shape and scale parameters, respectively. If set $\alpha = 1$ and $\sigma = 1$, F becomes standard normal distribution function.

2 The MLE of R

We want to estimate the R in which $X \sim SN(\alpha,\sigma)$ and $Y \sim SN(\beta,\sigma)$, and X and Y are independent random variables. Hence,

$$\begin{aligned} R &= P(Y < X) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{x} \frac{\alpha}{\sigma}\phi(\frac{x}{\sigma})[1-\Phi(\frac{x}{\sigma})]^{\alpha-1}.\frac{\beta}{\sigma}\phi(\frac{y}{\sigma})[1-\Phi(\frac{y}{\sigma})]^{\beta-1}dydx \\ &= 1-\int_{-\infty}^{+\infty} \frac{\alpha}{\sigma}\phi(\frac{x}{\sigma})[1-\Phi(\frac{x}{\sigma})]^{\alpha-1}[1-\Phi(\frac{x}{\sigma})]^{\beta}dx \end{aligned}$$

$$= \frac{\beta}{\alpha+\beta}. \tag{3}$$

Therefore, to estimate the R, we first evaluate the MLE of σ, α and β based on progressively Type-II censored data. Let $X_{1:m_1:n_1}, X_{2:m_1:n_1}, \ldots, X_{m_1:m_1:n_1}$ and $Y_{1:m_2:n_2}, Y_{2:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2}$ be two progressively Type-II censored samples from $SN(\alpha,\sigma)$ and $SN(\beta,\sigma)$ with censoring schemes $(R_1, R_2, ..., R_{m_1})$ and $(S_1, S_2, ..., S_{m_2})$, respectively. So, the likelihood function of the observed sample is given by ([2])

$$\begin{aligned} L(\alpha,\beta,\sigma) = &[c_1 \prod_{i=1}^{m_1} f(x_{i:m_1:n_1})[1-F(x_{i:m_1:n_1})]^{R_i}] \\ &\times [c_2 \prod_{i=1}^{m_2} f(y_{i:m_2:n_2})[1-F(y_{i:m_2:n_2})]^{S_i}], \end{aligned}$$

where

$$c_1 = n_1(n_1-R_1-1)\cdots(n_1-R_1-R_2-\cdots-R_{m_1-1}-m_1+1),$$

and

$$c_2 = n_2(n_2-S_1-1)\cdots(n_2-S_1-S_2-\cdots-S_{m_2-1}-m_2+1).$$

considering the $1-\Phi(a) = \Phi(-a)$, the log-likelihood function is given by

$$\begin{aligned} l(\alpha,\beta,\sigma) =& c-(m_1+m_2)\ln\sigma + m_1\ln\alpha + m_2\ln\beta + \sum_{i=1}^{m_1}\ln[\phi(\frac{x_{i:m_1:n_1}}{\sigma})] \\ &+ \sum_{i=1}^{m_2}\ln[\phi(\frac{y_{i:m_2:n_2}}{\sigma})] + \sum_{i=1}^{m_1}(\alpha R_i+\alpha-1)\ln[\Phi(-\frac{x_{i:m_1:n_1}}{\sigma})] \\ &+ \sum_{i=1}^{m_2}(\beta S_i+\beta-1)\ln[\Phi(-\frac{y_{i:m_2:n_2}}{\sigma})]. \end{aligned}$$

Hence, the MLEs of α and β, denote by $\hat{\alpha}_{(\hat{\sigma})}$ and $\hat{\beta}_{(\hat{\sigma})}$, are

$$\hat{\alpha}_{(\hat{\sigma})} = -\frac{m_1}{\sum_{i=1}^{m_1}(R_i+1)\ln[\Phi(-\frac{X_{i:m_1:n_1}}{\hat{\sigma}})]}, \tag{4}$$

$$\hat{\beta}_{(\hat{\sigma})} = -\frac{m_2}{\sum_{i=1}^{m_2}(S_i+1)\ln[\Phi(-\frac{Y_{i:m_2:n_2}}{\hat{\sigma}})]}, \tag{5}$$

and $\hat{\sigma}$ could be computed using the following non-linear equation via an iterative scheme

$$g(\sigma) = \sigma, \tag{6}$$

where

$$\begin{aligned} g(\sigma) = (m_1+m_2)^{-1}\Bigg[&\frac{1}{\sigma}\sum_{i=1}^{m_1} x_{i:m_1:n_1}^2 + \sum_{i=1}^{m_1}(\hat{\alpha}_{(\sigma)}R_i + \hat{\alpha}_{(\sigma)} - 1)x_{i:m_1:n_1} \\ &\times h(\frac{x_{i:m_1:n_1}}{\sigma}) + \frac{1}{\sigma}\sum_{i=1}^{m_2} y_{i:m_2:n_2}^2 + \sum_{i=1}^{m_2}(\hat{\beta}_{(\sigma)}S_i + \hat{\beta}_{(\sigma)} - 1)y_{i:m_2:n_2} \\ &\times h(\frac{y_{i:m_2:n_2}}{\sigma})\Bigg], \end{aligned} \tag{7}$$

and $h(t) = \frac{\phi(t)}{1-\Phi(t)}$ is the hazard rate function of the standard normal distribution.

So, from the invariant property of the ML estimators, the MLE of the R is evaluated as

$$\hat{R} = \frac{\hat{\beta}_{(\hat{\sigma})}}{\hat{\alpha}_{(\hat{\sigma})} + \hat{\beta}_{(\hat{\sigma})}}. \tag{8}$$

2.1 Bayesian estimation of R

We develop the Bayesian inference of R based on the Bayesian estimation of the parameters α, β and σ. We assume α, β and σ follow the conjugate independent gamma priors. Therefore,

$$\pi_i(\theta_i) = \frac{b_i^{a_i}}{\Gamma(a_i)}\theta_i^{a_i-1}e^{-b_i\theta_i}, \qquad \theta_i > 0, a_i > 0, b_i > 0, i = 1,2,3 \tag{9}$$

where $(\theta_1, \theta_2, \theta_3) = (\alpha, \beta, \sigma)$.

The joint density of the data, α, β and σ is

$$L(data, \alpha, \beta, \sigma) = L(data|\alpha, \beta, \sigma) \times \Pi_{i=1}^{3}\pi_i(\theta_i).$$

Therefore, the joint posterior density of α, β and σ given data can be written as

$$L(\alpha,\beta,\sigma|data)=\frac{L(data,\alpha,\beta,\sigma)}{\int_0^\infty\int_0^\infty\int_0^\infty L(data,\alpha,\beta,\sigma)d\alpha d\beta d\sigma}. \tag{10}$$

Since (10) cannot be determined in a closed form, therefore, to perform inference of the R, the MCMC methods are used. The posterior PDFs of α, β and σ are given in the following way:

$$\alpha|\beta,\sigma,data \sim \Gamma(a_1+m_1, b_1-\sum_{i=1}^{m_1}(R_i+1)\ln\Phi(-\frac{x_{i:m_1:n_1}}{\sigma})),$$

$$\beta|\alpha,\sigma,data \sim \Gamma(a_2+m_2, b_2-\sum_{i=1}^{m_2}(S_i+1)\ln\Phi(-\frac{y_{i:m_2:n_2}}{\sigma})),$$

and

$$\begin{aligned}\pi(\sigma|\alpha,\beta,data) \propto\ & \sigma^{a_3+m_1+m_2-1}e^{-b_3\sigma}\\ &\times\prod_{i=1}^{m_1}\phi(\frac{x_{i:m_1:n_1}}{\sigma})\prod_{i=1}^{m_1}(\Phi(-\frac{x_{i:m_1:n_1}}{\sigma}))^{\alpha R_i+\alpha-1}\\ &\times\prod_{i=1}^{m_2}\phi(\frac{y_{i:m_2:n_2}}{\sigma})\prod_{i=1}^{m_2}(\Phi(-\frac{y_{i:m_2:n_2}}{\sigma}))^{\beta S_i+\beta-1},\end{aligned} \tag{11}$$

where $\Gamma(.,.)$ stands for Gamma distribution.

The posterior PDF of σ is not known. Based on Figure 2, the normal distribution is a good proposal. Hence, we apply the normal proposal distribution to simulate this distribution using the Metropolis-Hasting method.

The posterior mean and variance of the R can evaluate as

$$\hat{E}(R|data)=\frac{1}{T}\sum_{j=1}^{T}R^{(j)}, \tag{12}$$

and

$$\hat{V}(R|data)=\frac{1}{T}\sum_{j=1}^{T}(R^{(j)}-\hat{E}(R|data))^2,$$

Figure 2 displays graph for a sequence of 1000 generations from posterior density functions of scale parameter.

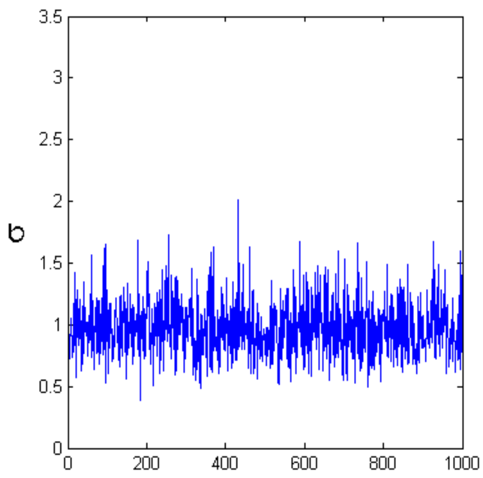

Figure 1: A sequence of 1000 generations from posterior density functions of σ

.

By choosing the shortest length from

$$\{(R_{(1)}, R_{([(1-\gamma)T])}), (R_{(2)}, R_{([(1-\gamma)T]+1)}) \dots, (R_{([\gamma T])}, R_{(T)})\}$$

, the HPD credible interval is determined, where $R_{(i)}$ is the ith order statistics from sample with T size.

3 Estimation of R with known scale parameter

The estimation of R is considered when the common scale parameter is known and equals 1.

3.1 The MLE of R

Considering to (8), the MLE of R will be

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} = \frac{1}{1 + \frac{m_1 W_2}{m_2 W_1}}, \tag{13}$$

where $W_1 = -\sum_{i=1}^{m_1}(R_i + 1)\ln(\Phi(-X_{i:m_1:n_1}))$ and $W_2 = -\sum_{i=1}^{m_2}(S_i + 1)\ln(\Phi(-Y_{i:m_2:n_2}))$.

Lemma 3.1. *Let $\{X_{1:m_1:n_1}, X_{2:m_1:n_1}, \dots, X_{m_1:m_1:n_1}\}$ be progressively Type-II censored samples from $SN(\alpha,\sigma)$ with censoring schemes $(R_1, R_2, \dots, R_{m_1})$. By*

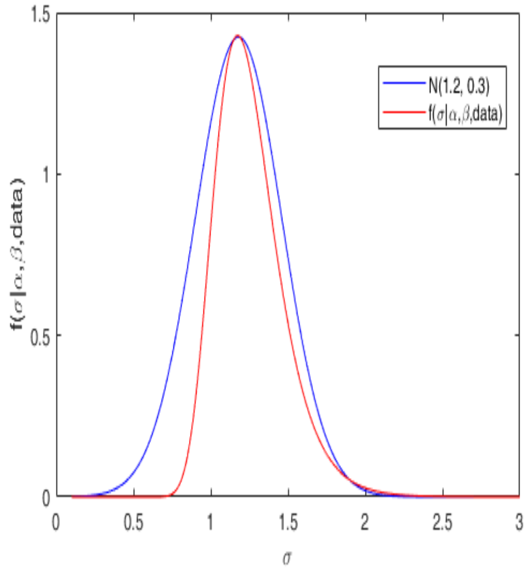

Figure 2: Proposal and posterior density functions of the scale parameter

defining

$$Z_1 = -n_1 \ln\Phi(-X_{1:m_1:n_1}),$$
$$Z_i = (n_1 - \sum_{k=1}^{i-1} R_k - i + 1)\Big[-\ln\Phi(-X_{i:m_1:n_1}) - (-\ln\Phi(-X_{i-1:m_1:n_1})\Big],$$
$$i = 1, 2, \dots, m_1,$$

distribution of Z_i*,* $i = 1, \dots, m_1$ *is exponential with mean* $\frac{1}{\alpha}$.

Proof. See Balakrishnan and Aggarwala [2]. □

Theorem 3.2. *Suppose that* $\{X_{1:m_1:n_1}, X_{2:m_1:n_1}, \dots, X_{m_1:m_1:n_1}\}$ *and* $\{Y_{1:m_2:n_2}, Y_{2:m_2:n_2}, \dots, Y_{m_2:m_2:n_2}\}$ *be two progressively Type-II censored samples from* $SN(\alpha, \sigma)$ *and* $SN(\beta, \sigma)$ *with censoring schemes* $(R_1, R_2, \dots, R_{m_1})$ *and* $(S_1, S_2, \dots, S_{m_2})$*, respectively. Then the* $100(1-\gamma)\%$ *CI for R is*

$$\left[\frac{1}{1 + (\frac{1}{\hat{R}} - 1) F_{2m_1, 2m_2; 1-\gamma/2}}, \frac{1}{1 + (\frac{1}{\hat{R}} - 1) F_{2m_1, 2m_2; \gamma/2}} \right] \tag{14}$$

where $F_{2m_1, 2m_1; \gamma/2}$ *and* $F_{2m_1, 2m_2; 1-\gamma/2}$ *are the lower and upper* $\frac{\gamma}{2}$*th percentile points of the F distribution with* $2m_1$ *and* $2m_2$ *degrees of freedom.*

Proof. From Lemma (3.1) it is evident that

$$W_1 = -\sum_{i=1}^{m_1}(R_i+1)\ln\Phi(-X_{i:m_1:n_1}) = \sum_{i=1}^{m_1} Z_i \sim \Gamma(m_1,\alpha),$$

and $2\alpha W_1$ is the chi-square random variable with $2m_1$ df parameter and likewise, $2\beta W_2$ is the chi-square random variable with $2m_2$ df. Thus by using

$$\hat{R} \sim \frac{1}{1+\frac{\alpha}{\beta}F} \quad \text{or} \quad \frac{R}{1-R}\times\frac{1-\hat{R}}{\hat{R}} \sim F_{2m_1,2m_2},$$

and

$$P(F_{2m_1,2m_2;\gamma/2} < \frac{R}{1-R}\times\frac{1-\hat{R}}{\hat{R}} < F_{2m_1,2m_2;1-\gamma/2}) = 1-\gamma.$$

the proof is completed. □

From Theorem (3.2), the PDF of $\hat{R}$ is resulted as:

$$f_{\hat{R}}(r) = \frac{1}{B(m_2,m_1)r^2}\left(\frac{m_2\beta}{m_1\alpha}\right)^{m_2}\frac{(\frac{1-r}{r})^{m_2-1}}{(1+\frac{m_2\beta}{m_1\alpha}(\frac{1-r}{r}))^{m_1+m_2}}, 0<r<1. \quad (15)$$

3.2 The Bayesian estimation of R

Since we assumed that the parameters are a priori independent with gamma density, the posterior density of α and β are independent $\Gamma(a_1+m_1,b_1+W_1)$ and $\Gamma(a_2+m_2,b_2+W_2)$, receptively. Therefore, the posterior disrtibution of R will be

$$\pi(R|data) = c.r^{a_1+m_1-1}(1-r)^{a_2+m_2-1}(1-zr)^{-(a_1+a_2+m_1+m_2)}, \quad (16)$$

where $0<r<1$,

$$c = \frac{\Gamma(a_1+a_2+m_1+m_2)}{\Gamma(a_1+m_1)\Gamma(a_2+m_2)}.\left(\frac{b_2+W_2}{b_1+W_1}\right)^{a_2+m_2} \quad (17)$$

and $z = 1-\frac{b_2+W_2}{b_1+W_1}$.

By assuming the quadratic loss function, the Bayesian estimation will be the posterior mean which could be computed by considering the following well-known equation

$$B(b,c-b)\,{}_2F_1(a,b;c;z) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx c>b>0, \quad (18)$$

in which $B(b,c-b)$ and ${}_2F_1(a,b;c;z)$ are beta and hypergeometric functions, receptively.

Therefore, the Bayesian estimation of R is

$$\hat{R}_{Bayes} = E(R|data) = \frac{\Gamma(a_1+a_2+m_1+m_2)}{\Gamma(a_1+m_1)\Gamma(a_2+m_2)} \left(\frac{b_2+W_2}{b_1+W_1}\right)^{a_2+m_2} B(a_1+m_1+1, a_2+m) \times F_1\left(q, a_1+m_1+1; a_1+a_2+m_1+m_2+1; 1-\frac{b_2+W_2}{b_1+W_1}\right), \quad (19)$$

where $q = a_1+a_2+m_1+m_2$. The variance of the Bayesian estimator could be achieved by using

$$E(R^2|data) = \frac{\Gamma(a_1+a_2+m_1+m_2)}{\Gamma(a_1+m_1)\Gamma(a_2+m_2)} . \left(\frac{b_2+W_2}{b_1+W_1}\right)^{a_2+m_2} B(a_1+m_1+2, a_2+m) \times {}_2F_1\left(p, a_1+m_1+2; a_1+a_2+m_1+m_2+2; 1-\frac{b_2+W_2}{b_1+W_1}\right),$$

where $p = a_1+a_2+m_1+m_2$. To construct the HPD intervals, as the posterior is not tractable, we can generate a sample from the posterior using indirect sampling algorithm, such as the accept-reject method.

4 Data Analysis

Two real strength data reported by Badar & Priest [3] are analyzed. We fitted the SN distribution models for two datasets separately. We also applied the Kolmogorov-Smirnov (K-S) test for two datasets that result reported in Table 1. According to the results, SN distribution fits well to both datasets. Moreover, Figures 3 confirms the appropriate fit. According to Table 1, it is clear that the scale parameters of two datasets are almost the same, Assuming equality, $\hat{R} = 0.5766$ and the related CI of R is $(0.4964, 0.6568)$.

Table 1: The Kolmogorov-Smirnov test for SN distributions on the real datasets.

Dataset	$\hat{\sigma}$	$\hat{\alpha}$	K-S	p-value
1	0.5962	0.8756	0.1013	0.5056
2	0.5096	1.0727	0.0412	0.9995

Two progressively censored schemes of Babayi and Khorram [4] are used to choose sample from the real data. Using equations (2) and (12), the MLE and Bayesian estimation of the R are 0.5530 and 0.5582, respectively. According to Congdon [5] and Kundu and Gupta [6], we set $P1$ to compute the Bayesian estimation. The 95% CIs based on the MLE and Bayesian estimation are $(0.3450, 0.7610)$ and $(0.3547, 0.7428)$, respectively. The 95% percentile bootstrap ([7]) and bootstrap-t methods ([8]) CIs are $(0.3319, 0.7463)$ and $(0.3407, 0.7513)$, respectively. The results show that inference based on the progressively censored dataset is not significantly different from the completed dataset.

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Stochastic Comparisons of Series Systems with Kumaraswamy-G Distributed Components having Archimedean Copulas

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Abstract: In this paper, we investigate the ordering properties of order statistics from dependent observations. We obtain the usual stochastic order for the smallest order statistic of samples having Kumaraswamy generalized family and Archimedean survival copulas. Some examples are provided to illustrate the established results.

Keywords: Archimedean copula, Kumaraswamy-G distribution, Majorization, Series systems, Usual stochastic order.

1 Introduction

Series and parallel systems are two basic systems which play prominent roles in various applications in reliability engineering. An n-component system with series (parallel) structure fails (works) if at least one of the components of the system fails (works). Let $X_1, X_2, ..., X_n$ denote the lifetimes of n components that can be used to built up an n component system. If $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the ordered lifetimes of the components then it is known that $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. Reliability and stochastic properties of series and parallel systems have been considered by various researchers under different scenarios. For example, stochastic comparisons of the lifetimes of series and parallel systems, in the case of

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heterogeneous component lifetimes with with exponentiated Weibull (EW) distributions, are considered in [4] and [8] and by [2] in the case of heterogeneous components with generalized exponential (GE) distributions. For comprehensive references one may refer to [6] and [1].

The paper by Kumaraswamy [7] proposed Kumaraswamy's distribution (Kw distribution) on $(0,1)$. The cumulative distribution function of a two-parameter Kw distribution with parameters (δ,γ), written as Kw(δ,γ), is given by

$$F(x) = 1-(1-x^{\delta})^{\gamma}, \quad 0<x<1, \delta>0, \gamma>0, \tag{1}$$

where δ and γ are the shape parameters. Generalizing this distribution, Cordeiro and de Castro [3] have proposed a new family of generalized distributions, called Kumaraswamy generalized family of distributions (called Kw-G distribution). The distribution function of the Kw-G random variable is represented as

$$F(x) = 1-(1-(G(x))^{\delta})^{\gamma}, \quad x>0, \delta>0, \gamma>0, \tag{2}$$

For convenience, henceforth, we denote $\mathrm{Kw}-\mathrm{G}(\delta,\gamma)$. Note that if $G(x)=x$, 2 corresponds to the distribution function of Kw distribution. For detailed discussion on this distribution, one may see [3]. Recently, Kundu and Chowdhury [9] studied some ordering properties of sample minimum from a Kw-G family of distributions. For maximums from independent and heterogeneous Kw-G samples, Kundu and Chowdhury [10] studied the ordering properties under random shock. Holding the assumption of independence, Kayal [5] studied stochastic comparisons of the series and parallel systems comprising Kw-G family of distributions.

By removing the condition of independence, this paper is devoted to further investigating how heterogeneity of the sample impact order statistics. We study the smallest order statistics from two dependent samples with Kw-G family of distributions. We derive the usual stochastic order of the smallest order statistics.

The organization of the paper is laid out as follows: Section 2 introduces the required definitions, and Section 3 presents several useful lemmas which are used throughout the paper and section 4 studies the usual stochastic order of the smallest order statistics from two Kw-G samples with Archimedean survival copulas. Finally Section 5 concludes the paper.

2 Preliminaries

There are many ways in which a random variable X can be said to be smaller than another random variable Y. In the usual stochastic ordering case, a random variable X with survival function $\bar{F} = 1 - F$ is stochastically smaller than a random variable Y with survival function $\bar{G} = 1 - G$, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all x. For more details on various kinds of stochastic orders, one may refer to [15].

For a random vector $X = (X_1, \ldots, X_n)$ with the joint distribution function F, joint survival function $\bar{F}$ and univariate survival functions $\bar{F}_1, \ldots, \bar{F}_n$, if there exists some $\hat{C} : [0,1]^n \longrightarrow [0,1]$ such that, for all x_i , $i = 1, \ldots, n$,

$$\bar{F}(x_1, \ldots, x_n) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)),$$

then $\hat{C}$ is called as the survival copula of X. A real function ϕ is n-monotone on $(a,b) \subseteq \mathbb{R}$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in (a,b) and $(-1)^k\phi^{(k)}(x) \geq 0$ for all $x \in (a,b), k = 0, 1, \ldots, n-2$, in which $\phi^{(i)}(.)$ is the ith derivative of $\phi(.)$. For a n-monotone $(n \geq 2)$ function $\phi : [0, +\infty) \longrightarrow [0,1]$ with $\phi(0) = 1$ and $\lim_{x \to +\infty} \phi(x) = 0$, let $\psi = \phi,^{-1}$ be the right continuous inverse of ψ, then

$$C_\phi(u_1, \ldots, u_n) = \phi(\psi(u_1) + \ldots + \psi(u_n)), \quad \text{for all}\, u_i \in [0,1], i = 1, \ldots, n,$$

is called an Archimedean copula with generator ϕ. Archimedean copulas cover a wide range of dependence structures including the independence copula. For more detail on Archimedean copulas, readers may refer to [14] and [13].

Majorization orders are quite useful and powerful in establishing various inequalities. For preliminary notations and terminologies on majorization theory, see [12]. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two real vectors anf

$x_{(1)} \leq \ldots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x}$.

Definition 2.1. The vector $\mathbf{x}$ is said to be

(i) weakly submajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_{\mathrm{w}} \mathbf{y}$) if $\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$ for all $j = 1, \ldots, n$,

(ii) weakly supermajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\mathrm{w}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n$,

(iii) majorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\mathrm{m}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n-1$.

It is well-known that

$$\mathbf{x} \overset{\mathrm{w}}{\preceq} \mathbf{y} \Longleftarrow \mathbf{x} \overset{\mathrm{m}}{\preceq} \mathbf{y} \Longrightarrow \mathbf{x} \preceq_{\mathrm{w}} \mathbf{y}, \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}_{+}^{n}.$$

The random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to follow the Kw-G distribution if X_i has the distribution function $F_i(x) = 1 - (1 - (G(x))^{\delta_i})^{\gamma_i}$ for $i = 1, \ldots, n$, where $G(x)$ is the baseline distribution function. Specifically, by $\mathbf{X} \sim \mathrm{Kw-G}(G, \delta, \gamma, \phi)$ we denote the sample having the Archimedean copula with generator ϕ and following a Kw-G model with baseline distribution function G.

3 Some useful lemmas

Before proceeding to main results, let us present some lemmas to be utilized in the sequel. The first two lemmas concern majorization, Schur-convexity and Schur-concavity.

Lemma 3.1 ([12], Theorem 3.A.4)**.** Suppose $\mathbb{I} \subset \mathbb{R}$ is an open interval and $\Phi : \mathbb{I}^n \longrightarrow \mathbb{R}_{+}$ is continuously differentiable. Necessary and sufficient conditions for Φ to be Schur-convex (Schur-concave) on $\mathbb{I}^n$ are

(i) Φ is symmetric on $\mathbb{I}^n$,

(ii) for $i \neq j$ and all $z \in \mathbb{I}^n$,

$$(z_i - z_j)\left(\frac{\partial\Phi(z)}{\partial z_i} - \frac{\partial\Phi(z)}{\partial z_j}\right) \geq (\leq)0,$$

where $\frac{\partial\Phi(z)}{\partial z_i}$ denotes the partial derivative of Φ with respect to its i-th argument.

Lemma 3.2 ([12], Theorem 3.A.8)**.** For a function l on $A \in \mathbb{R}^n$, $\mathbf{x} \preceq_{\mathrm{w}} (\overset{\mathrm{w}}{\preceq})\mathbf{y}$ implies $l(\mathbf{x}) \leq l(\mathbf{y})$ if and only if it is increasing (decreasing) and Schur-convex on A.

The following lower orthant order on Archimedean copulas will also be utilized in the sequel.

Lemma 3.3 ([11], Lemma A.1)**.** *For two n-dimensional Archimedean copulas $C_{\phi_1}(\mathbf{u})$ and $C_{\phi_2}(\mathbf{u})$, if $\psi_2 \circ \phi_1$ is super-additive, then $C_{\phi_1}(\mathbf{u}) \leq C_{\phi_2}(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^n$.*

Lemma 3.4 ([11], Lemma 3.4)**.** *For any $s \in [0,1]$, $J_1(\gamma, s, \phi) = \phi(\sum_{i=1}^n \psi(s^{\gamma_i}))$ is decreasing in γ_i for $i = 1, \ldots, n$. Furthermore, $J(\gamma, x, \phi)$ is Schur-concave (Schur-convex) with respect to γ whenever ϕ is log-convex (log-concave).*

4 On the smallest order statistic

This section studies the usual stochastic order on the smallest order statistic from the K*w*-G samples coupled by Archimedean survival copulas.

In the following theorem, we consider two minimum order statistics that are formed from two different sets of random variables having different sets of shape parameters δ but the same set of shape parameters γ.

Theorem 4.1. Suppose, for $X \sim \mathrm{Kw-G}(\delta, \gamma, \phi_1)$ and $X^* \sim \mathrm{Kw-G}(\delta^*, \gamma, \phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2 \circ \phi_1$ is super-additive, then $(\delta_1, \ldots, \delta_n) \overset{\mathrm{w}}{\succeq} (\delta_1^*, \ldots, \delta_n^*)$ implies $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$.

Proof. $X_{1:n}$ and $X^*_{1:n}$ have their respective survival functions, for $x \geq 0$,

$$\bar{F}_{X_{1:n}}(x) = p(X_k > x, 1 \leq k \leq n) = \phi(\sum_{i=1}^{n} \psi((1-(G(x))^{\delta_i})^{\gamma}))) = J(\delta, \gamma, x, \phi_1), \tag{3}$$

$$\bar{F}_{X^*_{1:n}}(x) = p(X_k > x, 1 \leq k \leq n) = \phi(\sum_{i=1}^{n} \psi((1-(G(x))^{\delta^*_i})^{\gamma}))) = J(\delta^*, \gamma, x, \phi_2). \tag{4}$$

We only prove the case that ϕ_1 is log-convex, and the other case can be finished similarly. The partial derivatives of $J(\delta, \gamma, x, \phi_1)$ with respect to δ_i are

$$\frac{\partial J(\delta, \gamma, x, \phi_1)}{\partial \delta_i} = -\gamma(1-(G(x))^{\delta_i})^{\gamma-1} \log(G(x))(G(x))^{\delta_i} \times$$

$$\frac{\phi'(\sum_{i=1}^{n} \psi((1-(G(x))^{\delta_i})^{\gamma})))}{\phi'(\psi((1-(G(x))^{\delta_i})^{\gamma})))} \geq 0,$$

$$\text{for all} \quad x > 0.$$

Thus $J(\delta, \gamma, x, \phi_1)$ is increasing with respect to δ_i's.

To prove its Schur-concavety, it follows from Lemma 3.1 that we have to show that for $i \neq j$,

$$(\delta_i - \delta_j)\left(\frac{\partial J(\delta, \gamma, x, \phi_1)}{\partial \delta_i} - \frac{\partial J(\delta, \gamma, x, \phi_1)}{\partial \delta_j}\right) \leq 0,$$

that is, for $i \neq j$,

$$-\gamma \log(G(x))\phi'(\sum_{i=1}^{n} \psi((1-(G(x))^{\delta_i})^{\gamma}))(\delta_i - \delta_j)$$

$$\left(\frac{(G(x))^{\delta_i}}{1-(G(x))^{\delta_i}} \frac{\phi_1(\psi_1((1-(G(x))^{\delta_i})^{\gamma}))}{\phi_1'(\psi_1((1-(G(x))^{\delta_i})^{\gamma}))} - \frac{(G(x))^{\delta_j}}{1-(G(x))^{\delta_j}} \frac{\phi_1(\psi_1((1-(G(x))^{\delta_j})^{\gamma}))}{\phi_1'(\psi_1((1-(G(x))^{\delta_j})^{\gamma}))}\right).$$

Note that the log-convexity of ϕ_1 implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Since $\psi_1((1-(G(x))^{\delta_i})^{\gamma})$ is decreasing in $\delta_i > 0$, then $\frac{\phi_1(\psi_1((1-(G(x))^{\delta_i})^{\gamma}))}{\phi_1'(\psi_1((1-(G(x))^{\delta_i})^{\gamma}))}$ is increasing in $\delta_i > 0$. Also the decreasing property of $\frac{(G(x))^{\delta_i}}{1-(G(x))^{\delta_i}}$ implies that

$\frac{(G(x))^{\delta_i}}{1-(G(x))^{\delta_i}} \frac{\phi_1(\psi_1((1-(G(x))^{\delta_i})^{\gamma}))}{\phi_1'(\psi_1((1-(G(x))^{\delta_i})^{\gamma}))}$ is increasing in $\delta_i > 0$. So, for $i \neq j$,

$$(\delta_i - \delta_j)\left(\frac{\partial J(\delta,\gamma,x,\phi_1)}{\partial \delta_i} - \frac{\partial J(\delta,\gamma,x,\phi_1)}{\partial \delta_j}\right) \leq 0,$$

Then Schur-concavety of $J(\delta,\gamma,x,\phi_1)$ follows from Lemma 3.1. According to Lemma 3.2 $(\delta_1,\ldots,\delta_n)\overset{\text{w}}{\succeq}(\delta_1^*,\ldots,\delta_n^*)$ implies $J(\delta,\gamma,x,\phi_1) \leq J(\delta^*,\gamma,x,\phi_1)$. On the other hand, since $\psi_2 \circ \phi_1$ is super-additive by Lemma 3.3, we have $J(\delta^*,\gamma,x,\phi_1) \leq J(\delta^*,\gamma,x,\phi_2)$. So, it holds that

$$J(\delta,\gamma,x,\phi_1) \leq J(\delta^*,\gamma,x,\phi_1) \leq J(\delta^*,\gamma,x,\phi_2).$$

That is, $X_{1:n} \leq_{\text{st}} X_{1:n}^*$. □

Example 4.2. Suppose that X and X^* have either of the following two dependence structures. (i) Gumbel survival copulas with respective generators

$$\phi_1(x) = e^{-x^{\frac{1}{\beta_1}}}, \quad \phi_2(x) = e^{-x^{\frac{1}{\beta_2}}}, \beta_2 \geq \beta_1 \geq 1;$$

(ii) Archimedean survival copulas with respective generators

$$\phi_1(x) = (x^{\frac{1}{\beta_1}})^{-1}, \quad \phi_1(x) = (x^{\frac{1}{\beta_1}})^{-1}, \beta_2 \geq \beta_1 \geq 1.$$

It is easy to see that ϕ_i is log-convex for $i = 1,2$. In view of $\psi_2(\phi_1(0)) = 0$ and the convexity of $\psi_2(\phi_1(x)) = x^{\frac{\beta_2}{\beta_1}}$, we conclude that $\psi_2(\phi_1(x))$ is super-additive by Proposition 21.A.11 in [12] .

Kayal [5] showed that if $X_1,\ldots,X_n$ be a set of independent random variables with $X_i \sim \text{Kw}-\text{G}(\delta_i,\gamma)$, $i = 1,\ldots,n$ and $X_1^*,\ldots,X_n^*$ be another set of independent random variables with $X_i^* \sim \text{Kw}-\text{G}(\delta_i^*,\gamma)$, $i = 1,\ldots,n$. Then

$$(\delta_1,\ldots,\delta_n)\overset{\text{w}}{\succeq}(\delta_1^*,\ldots,\delta_n^*) \quad \textit{implies} \quad X_{1:n} \leq_{\text{st}} X_{1:n}^*. \tag{5}$$

Theorem 1 partially improves the implication in 4 by relaxing the independence assumption in 4 under the Kw-G model.

Theorem 4.3. Suppose, for $X \sim \text{Kw}-\text{G}(\delta,\gamma,\phi_1)$ and $X^* \sim \text{Kw}-\text{G}(\delta^*,\gamma,\phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2 \circ \phi_1$ is super-additive, then $(\delta_1,\ldots,\delta_n)\overset{\text{m}}{\succeq}(\delta_1^*,\ldots,\delta_n^*)$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

Proof. We know that $(\delta_1,\ldots,\delta_n)\overset{\text{m}}{\succeq}(\delta_1^*,\ldots,\delta_n^*) \Longrightarrow (\delta_1,\ldots,\delta_n)\overset{\text{w}}{\succeq}(\delta_1^*,\ldots,\delta_n^*)$. Thus, the proof readily follows from Theorem 1 . □

In the next theorem we assume that two sets of random variables have the same set of shape parameters δ but different sets of shape parameters γ.

Theorem 4.4. Suppose, for $X \sim \mathrm{Kw-G}(\delta,\gamma,\phi_1)$ and $X^* \sim \mathrm{Kw-G}(\delta^*,\gamma^*,\phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2\circ\phi_1$ is super-additive, then $(\gamma_1,\ldots,\gamma_n)\succeq_{\text{w}}(\gamma_1^*,\ldots,\gamma_n^*)$ implies $X_{1:n}\leq_{\text{st}} X_{1:n}^*$.

Proof. $X_{1:n}$ and $X_{1:n}^*$ have their survival functions $J_1(\gamma,1-(G(x))^{\delta},\phi_1)$ and $J_1(\gamma,1-(G(x))^{\delta},\phi_2)$, respectively. Assume that ϕ_1 is log-convex. From Lemma 3.4, it follows that $-J_1(\gamma,1-(G(x))^{\delta},\phi_1)$ is increasing in γ_i for $i=1,\ldots,n$ and Schur-convex with respect to γ. According to Lemma 3.2, $(\gamma_1,\ldots,\gamma_n)\succeq_{\text{w}}(\gamma_1^*,\ldots,\gamma_n^*)$ implies $-J_1(\gamma,1-(G(x))^{\delta},\phi_1)\geq -J_1(\gamma^*,1-(G(x))^{\delta},\phi_1)$. On the other hand, since $\psi_2\circ\phi_1$ is super-additive by Lemma 3.3, we have $J_1(\gamma^*,1-(G(x))^{\delta},\phi_1)\leq J_1(\gamma^*,1-(G(x))^{\delta},\phi_2)$. So, it holds that $J_1(\gamma,1-(G(x))^{\delta},\phi_1)\leq J_1(\gamma^*,1-(G(x))^{\delta},\phi_1)\leq J_1(\gamma^*,1-(G(x))^{\delta},\phi_2)$. Assuming log-convex ϕ_2 instead, in a similar manner we can obtain $J_1(\gamma,1-(G(x))^{\delta},\phi_1)\leq J_1(\gamma,1-(G(x))^{\delta},\phi_2)\leq J_1(\gamma^*,1-(G(x))^{\delta},\phi_2)$. Both of the above two inequalities yield $X_{1:n}\leq_{\text{st}} X_{1:n}^*$. □

The next result immediately follows from Theorems 1 and 4.4.

Theorem 4.5. Suppose, for $X \sim \mathrm{Kw-G}(\delta,\gamma,\phi_1)$ and $X^* \sim \mathrm{Kw-G}(\delta^*,\gamma^*,\phi_2)$, ϕ_1 or ϕ_2 is log-convex and $\psi_2\circ\phi_1$ is super-additive, then $(\gamma_1,\ldots,\gamma_n)\succeq_{\text{w}}(\gamma_1^*,\ldots,\gamma_n^*)$ and $(\delta_1,\ldots,\delta_n)\overset{\text{w}}{\succeq}(\delta_1^*,\ldots,\delta_n^*)$ implies $X_{1:n}\leq_{\text{st}} X_{1:n}^*$.

The following theorem shows that we can get the ordering result in Theorem **??** under weakly supermajorization order.

Theorem 4.6. Suppose, for $X \sim \mathrm{Kw-G}(\delta,\gamma,\phi_1)$ and $X^* \sim \mathrm{Kw-G}(\delta,\gamma^*,\phi_2)$, ϕ_1 or ϕ_2 is log-concave and $\psi_1\circ\phi_2$ is super-additive, then $(\gamma_1,\ldots,\gamma_n)\overset{\text{w}}{\succeq}(\gamma_1^*,\ldots,\gamma_n^*)$ implies $X_{1:n}\geq_{\text{st}} X_{1:n}^*$.

Proof. $X_{1:n}$ and $X^*_{1:n}$ have their survival functions $J_1(\gamma, 1-(G(x))^\delta, \phi_1)$ and $J_1(\gamma, 1-(G(x))^\delta, \phi_2)$, respectively. Assume that ϕ_1 is log-concave. From Lemma 3.4, it follows that $J_1(\gamma, 1-(G(x))^\delta, \phi_1)$ is decreasing in γ_i for $i = 1,\ldots,n$ and Schur-convex with respect to γ. According to Lemma 3.2, $(\gamma_1,\ldots,\gamma_n) \overset{\text{w}}{\succeq} (\gamma_1^*,\ldots,\gamma_n^*)$ implies $J_1(\gamma, 1-(G(x))^\delta, \phi_1) \geq J_1(\gamma^*, 1-(G(x))^\delta, \phi_1)$. On the other hand, since $\psi_1 \circ \phi_2$ is super-additive by Lemma 3.3, we have $J_1(\gamma^*, 1-(G(x))^\delta, \phi_1) \geq J_1(\gamma^*, 1-(G(x))^\delta, \phi_2)$. So, it holds that $J_1(\gamma, 1-(G(x))^\delta, \phi_1) \geq J_1(\gamma^*, 1-(G(x))^\delta, \phi_1) \geq J_1(\gamma^*, 1-(G(x))^\delta, \phi_2)$. Assuming log-convex ϕ_2 instead, in a similar manner we can obtain $J_1(\gamma, 1-(G(x))^\delta, \phi_1) \geq J_1(\gamma, 1-(G(x))^\delta, \phi_2) \geq J_1(\gamma^*, 1-(G(x))^\delta, \phi_2)$. Both of the above two inequalities yield $X_{1:n} \geq_{\text{st}} X^*_{1:n}$. □

Example 4.7. Let X and X^* have the Gumbel-Hougaard survival copulas with respective generators

$$\phi_1(x) = e^{\frac{1}{\beta_1}(1-e^x)}, \quad \phi_2(x) = e^{\frac{1}{\beta_2}(1-e^x)}, 1 \geq \beta_2 \geq \beta_1 > 0,$$

It is easy to see that ϕ_i is log-concave for $i = 1,2$. Since $\psi_1(\phi_2(x))$is convex for $1 \geq \beta_2 \geq \beta_1 > 0$, we conclude that $\psi_1(\phi_2(x))$ is super-additive by Proposition 21.A.11 in [12] .

5 Concluding Remarks

This is the first try to study some stochastic comparisons of order statistics from dependent and heterogeneous samples having Kumaraswamy generalized family. We derived the usual stochastic order for the smallest order statistic of samples having Kumaraswamy generalized family and Archimedean survival copulas. Several examples are provided to illustrate the established results.

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Some Reliability Properties of Sequential $(n-r+1)$-out-of-n Systems

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Abstract: The hazard and reversed hazard rates are two important measures to study the lifetime random variables in reliability theory, survival analysis and stochastic modeling. In this article, we study the increasing hazard rate and decreasing reversed hazard rate and some other related properties of sequential $(n-r+1)$-out-of-n systems. Since we do not impose restrictions as previous studies did, our findings yield new results for various useful models of ordered random variables. Finally, some applications of these results are indicated.

Keywords: Aging properties, Record and Pfeifers record values, Sequential order statistics, Total positivity.

1 Introduction

A system with n independent components which functions if and only if at least $n-r+1$ of the components are working is called a $(n-r+1)$-out-of-n system. Parallel and series systems are particular cases of such systems corresponding to $r=n$ and $r=1$, respectively. These systems play an important role in reliability theory and life testing and in the real world. The lifetime of such a system is described by the r-th order statistic (OS) in a sample of size

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n when assuming that the remaining components are not affected by failures. More generally, the failure of one component may influence the remaining components. Thus, a more flexible model for a $(n-r+1)$-out-of-n system should take the dependence structure into consideration. The failure of some component of the system can more or less strongly influence the life-length distributions of the remaining components. This can be thought of as damage caused by the i-th failure in the system. For example, the breakdown of an aircrafts engine will increase the load put on the remaining engines, such that their lifetime should tend to be shorter. For dealing with this type of problem, sequential order statistics (SOS) are proposed in [9, 10]. It is worth mentioning that one of the most important and applicable case is SOS under proportional hazard rate assumption which have a one to one correspondence to the generalized order statistics (GOS) model introduced by [9, 10]. These models are closely connected to several other models of ordered random variables and, in particular they unify OS, progressively Type-II censored order statistics, record values, Pfeifer's record values, etc.

In the last three decades, a wide interest has been shown in investigating several aging properties of OS and other ordered random variables. One of the most important of these studies concerns the class of increasing hazard rate distributions (IHR). See e.g., [4], [18] and [15] for OS and [12] for record values. [9] generalized these results to GOS imposing some restrictions in his Chapter V. Also, some aging properties of SOS were obtained by [20]. In this article, obtaining useful lemmas and formulas, we prove that how some reliability properties transfers among SOS.

The article is organized as follows. In Section 2, we recall some definitions which is used in the paper including aging notions and the concept of SOS. In Section 3, we study some aging properties among SOS. Finally, in Section 4, we provide some applications in the submodels of SOS.

2 Definitions and notions

We recall some definitions and well known notions which will be used in the sequel. The word increasing (decreasing) is used for non-decreasing (non-increasing) and all expectations are implicitly assumed to exist whenever they are written.

2.1 Aging notions

For more details of the following concepts, we refer the reader to [11] and [4].

Definition 2.1. Let $\mathscr{X}$ and $\mathscr{Y}$ be subsets of the real line $\mathbb{R}$. A function $\Lambda : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ is said to be totally positive of order 2 (TP_2) (reverse regular of order 2 (RR_2)) if

$$\Lambda(x_1,y_1)\Lambda(x_2,y_2) - \Lambda(x_1,y_2)\Lambda(x_2,y_1) \geq (\leq) 0,$$

for all $x_1 \leq x_2$ in $\mathscr{X}$ and $y_1 \leq y_2$ in $\mathscr{Y}$ or, equivalently, if $\Lambda(x_2,y)/\Lambda(x_1,y)$ is increasing (decreasing) in y when $x_1 \leq x_2$.

Note that if the functions $\Lambda_1(x,y)$ and $\Lambda_2(x,y)$ are TP_2 (RR_2) in (x,y), then their product is TP_2 (RR_2) in (x,y) (cf. [11], p.123).

Now, let X be univariate random variable with cdf F, survival function (sf) $\bar{F} = 1 - F$, and probability density function (pdf) f, respectively. We denote the hazard rate of X by $h = f/\bar{F}$ and its reversed hazard rate by $r = f/F$. Let F^{-1} be the right continuous inverse (quantile function) of F.

Definition 2.2. The random variable X (or its distribution) is said to be
(i) increasing (decreasing) hazard rate, IHR (DHR), if $\bar{F}$ is logconcave (logconvex), or equivalently, $h(x)$ is increasing (decreasing) in x;
(ii) increasing (decreasing) reversed hazard rate, IRHR (DRHR), if F is logconvex (logconcave), or equivalently, $r(x)$ is increasing (decreasing) in x;
(iii) increasing (decreasing) hazard rate average, IHRA (DHRA), if $-\ln \bar{F}(x)$ is starshaped, i.e., $-\ln \bar{F}(x)/x$ is increasing in x (anti starshaped, i.e., $-\ln \bar{F}(x)/x$ is decreasing in x) when X is a non-negative random variable.

2.2 Sequential order statistics

As a generalization of OS and record values, SOS and GOS were introduced by [9, 10] as a unified approach to a variety of models of ordered random variables with different interpretations and statistical applications. Sequential order statistics are defined by means of a triangular scheme of random variables where the r-th line contains $n-r+1$ random variables with distribution function F_i, $i=1,...,n$.

Definition 2.3. Let $F_1,...,.F_n$ be continuous distribution functions with $F_1^{-1}(1)\leq ...\leq F_n^{-1}(1)$ and let $\{Y_{r,n}^{(j)},\ 1\leq j\leq n-r+1\}$ be a sequence of independent and identically distributed random variables each distributed according to F_r, where $r=1,...n$. Let $X_{1,n}^{(j)}=Y_{1,n}^{(j)}$, $1\leq j\leq n$, and denote $X_{1,n}^{*}=\min_{j=1}^{n}X_{1,n}^{(j)}$. For $r=2,...,n$, define $X_{r,n}^{(j)}=F_r^{-1}\{F_r(Y_{r,n}^{(j)})[1-F_r(X_{r-1,n}^{*})]+F_r(X_{r-1,n}^{*})\}$ and denote $X_{r,n}^{*}=\min_{j=1}^{n-r+1}X_{r,n}^{(j)}$. Then, $X_{1,n}^{*},...,X_{n,n}^{*}$ are called SOS based on $\{F_1,...,F_n\}$

From now on we cosider a particular choice of the distribution functions $F_1,...,F_n$, namely

$$F_r(t)=1-(1-F(t))^{\alpha_r},\ r=1,...,n\ ,$$

with some distribution function F and positive real numbers $\alpha_1,...,\alpha_n$. This is usually referred as the proportional hazard rate assumption (see, e.g., [16] and [8], for new extensions of proportional hazard rate model).

In this case, there exist several representations for the marginal pdf of SOS (see, e.g., [6]). Suppose that $n\in\mathbb{N}$, $m_i=(n-i+1)\alpha_i-(n-i)\alpha_{i+1}-1\in\mathbb{R}$, $i=1,...,n-1$, $\widetilde{m}_n=(m_1,...,m_{n-1})$, if $n\geq 2$ ($\widetilde{m}_n\in\mathbb{R}$ is arbitrary, if $n=1$) and $\gamma_r(n,\widetilde{m}_n)=(n-r+1)\alpha_r$ for $r=1,...,n$. In this case, we denote r-th SOS by $X_{(r,n,\widetilde{m}_n)}$. [7] obtained the expression

$$f_{X_{(r,n,\widetilde{m}_n)}}(x)=c_{r-1}(n,\widetilde{m}_n)[\bar{F}(x)]^{\gamma_r(n,\widetilde{m}_n)-1}g_r(F(x))f(x),\quad x\in\mathbb{R},\tag{1}$$

where $c_{r-1}(n,\widetilde{m}_n)=\prod_{i=1}^{r}\gamma_i(n,\widetilde{m}_n)$, $r=1,...,n$, and g_r is a particular Meijer's G-function. [19] rediscovered this representation by presenting an integral representation for g_r. According to Lemma 2.1 of Alimohammadi and Alamatsaz

[1], we have the following recursive formula:

$$g_1(u) \equiv 1,\ g_r(u) = \int_0^u g_{r-1}(t)[1-t]^{m_{r-1}}dt,\ 0 \le u \le 1,\ r = 2,\ldots,n. \tag{2}$$

For example, if $\alpha_i = 1$, $\forall i$, or $\alpha_i = \frac{1}{n-i+1}$, $\forall i$, then the SOS would convert to the OS and record values, respectively (see Table 1 of [9]).

3 Main results

We need some properties for the marginal pdf of SOS. Alimohammadi and Alamatsaz [1], Alimohammadi et al. [2, 3] studied logconcavity properties of the function g_r and GOS.

We first recall the following theorem known as the variation diminishing property.

Theorem 3.1 ([11]). *Let $\Lambda : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ be a TP_2 Borel-measurable function, $d\sigma(y)$ be a sigma-finite measure defined on $\mathscr{Y}$, and $\Lambda_1 : \mathscr{Y} \to \mathbb{R}$ be a bounded measurable function such that the integral $\Lambda_2(x) = \int_{\mathscr{Y}} \Lambda(x,y)\Lambda_1(y)d\sigma(y)$ converges absolutely. If Λ_1 changes sign at most j times on $\mathscr{Y}$ and $j \le 1$, then Λ_2 changes sign at most j times on $\mathscr{X}$. Moreover, if Λ_2 changes sign j times, then it must have the same arrangement of signs as does Λ_1.*

Lemma 3.2. *Let $g_{r'}(u) = \int_0^u g_{r'-1}(t)[1-t]^{m'_{r'-1}}dt,\ 0 \le u \le 1,\ r' = 2,\ldots,n.$ If $r \le r'$ and $m'_{r'-i} \le m_{r-i}$ for $1 \le i \le r-1$, then $g_{r'}(u)/g_r(u)$ is increasing in u.*

Proof. The ratio $g_{r'}(u)/g_r(u)$ is increasing if, and only if, for every $c > 0$, the function $g_{r'}(u) - cg_r(u)$ has at most one change of sign, and if so, it would be from $-$ to $+$. From (2), we have

$$\begin{aligned} g_{r'}(u) - cg_r(u) &= \int_0^u g_{r'-1}(t)[1-t]^{m'_{r'-1}}dt - c\int_0^u g_{r-1}(t)[1-t]^{m_{r-1}}dt \\ &= \int_{\mathbb{R}} \Lambda(u,t)\Lambda_1(t)dt, \end{aligned}$$

where

$$\Lambda(u,t) = I_{\{0\le t\le u\}}, \Lambda_1(t) = g_{r-1}(t)[1-t]^{m_{r-1}}\left(\frac{g_{r'-1}(t)}{g_{r-1}(t)}\frac{[1-t]^{m'_{r'-1}}}{[1-t]^{m_{r-1}}} - c\right). \tag{3}$$

Obviously, $g_{r'}(u)/g_r(u)$ is increasing for $r=1$ and $r'\geq 1$. Furthermore, $\frac{g_{r'-1}(t)}{g_{r-1}(t)}$ is increasing in t for $r=2$ and $r'\geq 2$. Because of $m'_{r'-i}\leq m_{r-i}$ for $2\leq i\leq r-1$, now suppose that $\frac{g_{r'-1}(t)}{g_{r-1}(t)}$ is increasing in t. Since $m'_{r'-1}\leq m_{r-1}$, the expression in parentheses of $\Lambda_1(t)$ changes sign at most once, and if once from $-$ to $+$. Also, $\Lambda(u,t)$ is TP_2. Therefore, the result follows using induction and Theorem 3.1. □

Lemma 3.3. *The survival function of $X_{(r,n,\widetilde{m}_n)}$ is given by*

$$P(X_{(r,n,\widetilde{m}_n)}>x)=\sum_{j=1}^{r}\frac{c_{r-1}(n,\widetilde{m}_n)}{\prod_{i=j}^{r}\gamma_i(n,\widetilde{m}_n)}[\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)}g_j(F(x)),\ x\in\mathbb{R}. \tag{4}$$

Proof. Using integration by parts and equations (2), (1) as well as the fact $m_{i-1}+\gamma_i(n,\widetilde{m}_n)=\gamma_{i-1}(n,\widetilde{m}_n)-1$, we obtain

$$\begin{aligned}
& P(X_{(r,n,\widetilde{m})}>x)\\
=\ & \frac{c_{r-1}(n,\widetilde{m}_n)}{\gamma_r(n,\widetilde{m}_n)}[\bar{F}(x)]^{\gamma_r(n,\widetilde{m}_n)}g_r(F(x))\\
& +\int_x^{\infty}\frac{c_{r-1}(n,\widetilde{m}_n)}{\gamma_r(n,\widetilde{m}_n)}[\bar{F}(y)]^{\gamma_{r-1}(n,\widetilde{m}_n)-1}g_{r-1}(F(y))f(y)dy\\
\vdots\ & \\
=\ & \sum_{j=2}^{r}\frac{c_{r-1}(n,\widetilde{m}_n)}{\prod_{i=j}^{r}\gamma_i(n,\widetilde{m}_n)}[\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)}g_j(F(x))\\
& +\int_x^{\infty}\frac{c_{r-1}(n,\widetilde{m}_n)}{\prod_{i=2}^{r}\gamma_i(n,\widetilde{m}_n)}[\bar{F}(y)]^{\gamma_1(n,\widetilde{m}_n)-1}f(y)dy.
\end{aligned}$$

This yields (4). □

Thus, the hazard rate of $X_{(r,n,\widetilde{m}_n)}$, the r-th SOS, is given by

$$h_{X_{(r,n,\widetilde{m}_n)}}(x)=\frac{f(x)}{\bar{F}(x)}\frac{[\bar{F}(x)]^{\gamma_r(n,\widetilde{m}_n)}g_r(F(x))}{\sum_{j=1}^{r}\frac{1}{\prod_{i=j}^{r}\gamma_i(n,\widetilde{m}_n)}[\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)}g_j(F(x))}. \tag{5}$$

Without the restriction $m_1=m_2=...=m_{n-1}$, [6] showed that if X is IHR, then so is $X_{(r,n,\widetilde{m}_n)}$, for any $r\in\{1,\ldots,n\}$ (see also Theorem 8 of [20]). From now on, we extend this result. We need to recall the following theorem known as the basic composition formula.

Theorem 3.4 ([11]). *Let $\Lambda_1 : \mathscr{X} \times \mathscr{Z} \to \mathbb{R}$, $\Lambda_2 : \mathscr{Z} \times \mathscr{Y} \to \mathbb{R}$ and $\Lambda : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ be Borel-measurable functions of two variables satisfying*

$$\Lambda(x,y) = \int_{\mathscr{Z}} \Lambda_1(x,z)\Lambda_2(z,y)d\sigma(z),$$

where $d\sigma(z)$ denotes a sigma-finite measure defined on $\mathscr{Z}$. If one of the functions Λ_1 or Λ_2 is RR_2 and the other TP_2, then Λ is RR_2 and, otherwise, if both of the them are RR_2 or TP_2, then Λ is TP_2.

We are now ready to provide main results of this section.

Theorem 3.5. *If $X_{(r,n,\widetilde{m}_n)}$ is IHR and $m_i \geq -1$, $\forall i$, then $X_{(r+1,n,\widetilde{m}_n)}$ is IHR.*

Proof. From (5) we have

$$\frac{h_{X_{(r+1,n,\widetilde{m}_n)}}(x)}{h_{X_{(r,n,\widetilde{m}_n)}}(x)} = \frac{\Lambda(r,x)}{\Lambda(r+1,x)} \frac{g_{r+1}(F(x))}{g_r(F(x))} [\bar{F}(x)]^{\gamma_{r+1}(n,\widetilde{m}_n)-\gamma_r(n,\widetilde{m}_n)}, \tag{6}$$

where

$$\Lambda(r,x) = \sum_{j=1}^{\infty} \frac{1}{\prod_{i=j}^{r} \gamma_i(n,\widetilde{m}_n)} [\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)} g_j(F(x)) I_{\{1\leq j\leq r\}},$$

and I_A is the indicator function. Note that $[\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)}$ is RR_2 in $(x,j) \in \mathbb{R} \times \{1,...,r\}$ and $1/\prod_{i=j}^{r} \gamma_i(n,\widetilde{m}_n)$ is TP_2 in $(j,r) \in \{1,...,r\} \times \{1,...,n\}$ bccause of $m_i \geq -1$. According to Lemma 3.2, $g_j(F(x))$ is TP_2 in $(x,j) \in \mathbb{R} \times \{1,...,r\}$ for $\widetilde{m}_n \in \mathbb{R}^{n-1}$ and $I_{\{1\leq j\leq r\}}$ is TP_2 in $(j,r) \in \{1,...,r\} \times \{1,...,n\}$. Thus, by Theorem 3.4, it follows that $\Lambda(r,x)$ is RR_2 in $(r,x) \in \{1,...,n\} \times \mathbb{R}$ and, hence, the first fraction in (6) is increasing in x. Again, Lemma 3.2 implies that the second fraction is increasing in x. Furthermore, the third factor is increasing bccause of $m_i \geq -1$. So, $h_{X_{(r+1,n,\widetilde{m}_n)}}(x)$ increases because $h_{X_{(r,n,\widetilde{m}_n)}}(x)$ is increasing. □

[11] generalized Theorem 3.4 as follows. If $\Lambda_1(x,y,z) > 0$ is TP_2 in each pairs of variables when the third variable is held fixed and $\Lambda_2(z,y)$ is TP_2, then $\Lambda(x,y) = \int_{\mathscr{Z}} \Lambda_1(x,y,z)\Lambda_2(z,y)d\sigma(z)$, is TP_2. He also note that the result is valid if $\Lambda_2(x,z,y) > 0$ is also a function of three variables. But, $\Lambda_2(x,z,y)$ must be remained TP_2 in each pairs of variables. We extend these findings

while our method of proof is completely different from the rather involved approach of [11]. Indeed, the idea of our method comes from the Lemma 2.2 of [14] that is often used in establishing the monotonicity of a fraction in which the numerator and denominator are integrals or summations.

Theorem 3.6. *Let $\Lambda_1 : \mathscr{X} \times \mathscr{Y} \times \mathscr{Z} \to \mathbb{R}_+$, $\Lambda_2 : \mathscr{X} \times \mathscr{Y} \times \mathscr{Z} \to \mathbb{R}_+$ and $\Lambda : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}_+$ be Borel-measurable functions satisfying*

$$\Lambda(x,y) = \int_{\mathscr{Z}} \Lambda_1(x,y,z)\Lambda_2(x,y,z)d\sigma(z), \tag{7}$$

where $d\sigma(z)$ denotes a sigma-finite measure defined on $\mathscr{Z}$.
(i) If Λ_1 and Λ_2 are RR_2 in (y,z) and (x,z), and, Λ_1 and Λ_2 are TP_2 in (x,y), then Λ is TP_2 in (x,y);
(ii) (a) If Λ_1 and Λ_2 are RR_2 in (y,z) and (x,y), and, Λ_1 and Λ_2 are TP_2 in (x,z),
or,
(b) if Λ_1 and Λ_2 are RR_2 in (x,y) and (x,z), and, Λ_1 and Λ_2 are TP_2 in (y,z), then Λ is RR_2 in (x,y).

Proof. We just prove part (i) and part (ii) follows similarly. We denote the expectation of X with respect to the pdf l_i, $i = 1,2$, by $E_{l_i}[X]$. For $x_1 \leq x_2$, from (7) we have

$$\frac{\Lambda(x_2,y_2)}{\Lambda(x_1,y_2)} = E_{l_2}[\varphi(x_1,x_2,y_2,Z)],$$

where

$$\varphi(x_1,x_2,y_i,z) = \frac{\Lambda_1(x_2,y_i,z)\Lambda_2(x_2,y_i,z)}{\Lambda_1(x_1,y_i,z)\Lambda_2(x_1,y_i,z)}, \; i = 1,2,$$

and, $Z|y_i$ is a random variable with respect to σ and the corresponding pdf

$$l_i(z|y_i) = c \cdot \Lambda_1(x_1,y_i,z)\Lambda_2(x_1,y_i,z), \; z \in \mathbb{R}, \; i = 1,2,$$

where $c = \left[\int_{\mathscr{Z}} \Lambda_1(x_1,y_i,z)\Lambda_2(x_1,y_i,z)d\sigma(z)\right]^{-1}$ is the normalizing constant. Since Λ_1 and Λ_2 are RR_2 in (y,z), we have $Z|y_1 \geq_{lr} Z|y_2$ and hence $Z|y_1 \geq_{st} Z|y_2$ whenever $y_1 \leq y_2$. Furthermore, Since Λ_1 and Λ_2 are RR_2 in (x,z),

$\varphi(x_1,x_2,y_2,z)$ is decreasing in z. Thus, because of $X \leq_{st} Y \Leftrightarrow E[\varphi(X)] \geq (\leq)E[\varphi(X)]$ for all decreasing (increasing) functions φ (cf. [17]), we have

$$E_{l_2}[\varphi(x_1,x_2,y_2,Z)] \geq E_{l_1}[\varphi(x_1,x_2,y_2,Z)].$$

Now, since Λ_1 and Λ_2 are TP_2 in (x,y), for $y_1 \leq y_2$ we have

$$E_{l_1}[\varphi(x_1,x_2,y_2,Z)] \geq E_{l_1}[\varphi(x_1,x_2,y_1,Z)] = \frac{\Lambda(x_2,y_1)}{\Lambda(x_1,y_1)}.$$

As required. □

Theorem 3.7. *Let $X_{(r,n,\widetilde{m}_n)}$ be IHR.*
(i) If $m_i \geq -1$, $\forall i$, then $X_{(r,n-1,\widetilde{m}_n)}$ is IHR;
(ii) If $m_i < -1$, $\forall i$, then $X_{(r,n+1,\widetilde{m}_n)}$ is IHR.

Proof. (i) From (5) we have

$$\frac{h_{X_{(r,n-1,\widetilde{m}_n)}}(x)}{h_{X_{(r,n,\widetilde{m}_n)}}(x)} = \frac{\Lambda(n,x)}{\Lambda(n-1,x)}[\bar{F}(x)]^{\gamma_r(n-1,\widetilde{m}_n)-\gamma_r(n,\widetilde{m}_n)},$$

where

$$\Lambda(n,x) = \sum_{j=1}^{r} \frac{1}{\prod_{i=j}^{r}\gamma_i(n,\widetilde{m}_n)}[\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)}g_j(F(x)).$$

It is not difficult to see that $[\bar{F}(x)]^{\gamma_j(n,\widetilde{m}_n)}$ and $\frac{g_j(F(x))}{\prod_{i=j}^{r}\gamma_i(n,\widetilde{m}_n)}$, as two functions of three variables (x,n,j), satisfy the conditions on Λ_1 and Λ_2 in part (i) of Theorem 3.6. Thus, the rest of the proof follows similar to that of Theorem 3.5.
(ii) This part follows similarly by means of part (ii) of Theorem 3.6. □

Remark 3.8. [9] obtained Theorems 3.5 and 3.7 (which also contain previous findings of OS and record values) for $m_1 = m_2 = \ldots = m_{n-1}$.

At next, we give the following result concerning transmission on parameters $\widetilde{m}_n$. The proof is similar to those of above theorems.

Theorem 3.9. *Let $m'_i = (n-i+1)\alpha'_i - (n-i)\alpha'_{i+1} - 1$ and $\widetilde{m'}_n = \{m'_1,\ldots,m'_{n-1}\}$. If $X_{(r,n,\widetilde{m}_n)}$ is IHR and $m_i \geq m'_i$, $\forall i$, then $X_{(r,n,\widetilde{m'}_n)}$ is IHR.*

Remark 3.10. The results for the transmission of DHR property is proved similarly in the reverse direction.

At the end of this section, we examine the transmission of the DRHR property among SOS. Such a study was started by [13] for OS and record values. Then, [21] generalized these results for GOS under the restriction $m_1 = m_2 = \ldots = m_{n-1}$. For brevity, we omit the proofs.

Theorem 3.11. *Let $m_i \geq 0$, $1 \leq i \leq r-1$, and $\gamma_r(n,\widetilde{m}_n) \geq 1$, $1 \leq r \leq n$. If X is DRHR, then $X_{(r,n,\widetilde{m}_n)}$ is DRHR.*

Theorem 3.12. *Let $m_i \geq 0$, $1 \leq i \leq r-1$, $\gamma_r(n,\widetilde{m}_n) \geq 1$, $1 \leq r \leq n$, $\widetilde{m'}_n = \{m'_1, \ldots, m'_{n-1}\}$ and $m'_i \geq m_i$. If $X_{(r,n,\widetilde{m}_n)}$ is DRHR, then $X_{(r-1,n,\widetilde{m}_n)}$, $X_{(r,n+1,\widetilde{m}_n)}$ and $X_{(r,n,\widetilde{m'}_n)}$ are DRHR.*

Remark 3.13. According to Remark 1.16 of [9], the main IHR results of this section can be used for transmission of the IHRA, NBU (new better than used) and DMRL (decreasing mean residual life) properties among SOS (and, analogously, for transmission of the DHRA, NWU (new worse than used) and IMRL (increasing mean residual life) properties in the reverse direction). Also, according to Lemma 2.3 of [13], Theorem 3.12 is valid for IUPL (increasing uncertainty in past life) property.

4 Applications

We note that the results of this paper can be applied for the submodels of SOS and GOS with unequal m_i as well as with equal m_i. We give just two examples.

Ordinary record and Pfeifer's record values. The record values were defined as a model for successive extremes in a sequence of iid random variables. They are closely connected with the occurrence times of non-homogeneaus Poisson process, shocks models and minimal repair. Pfeifer's record model is based on non-identically distributed random variables and, thus, ordinary record values are contained in Pfeifer's model. In this model, the distribution of the underlying random variables may change after each record event. A particular choice of distribution functions

$$F_i(x) = 1 - (1 - F(x))^{\beta_i},\ x \in \mathbb{R},\ 1 \leq i \leq n,$$

with a cdf F and $\beta_i > 0$ leads to the model of SOS with parameters $\alpha_i = \frac{\beta_i}{n-i+1}$ (cf. [9]). According to Theorem 3.5, if the r-th Pfeifer's record value is IHR (IHRA, NBU or DMRL), then so does the $r+1$-th one provided that $\beta_i \geq \beta_{i+1}$. Notice that this condition particularly holds for ordinary record values because of $\beta_i = 1,\ \forall i$.

Sequential $(n-r+1)$-out-of-n system. The r-th SOS can be used to describe the lifetime of a sequential $(n-r+1)$-out-of-n system (cf. [5]). According to part (i) of Theorem 3.7, if a sequential $(n-r+1)$-out-of-n system is IHR (IHRA, NBU or DMRL), then a sequential parallel system with r components is also IHR (IHRA, NBU or DMRL) provided that $(r-i+1)\alpha_i \geq (r-i)\alpha_{i+1}$, for $i = 1,..,r-1$. Note that this system is parallel and would have fewer components than the initial system.

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On a New Extension of Weibull Distribution with Application to Lifetime Data

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Abstract: Any given system can be represented as a parallel arrangement of series structures. Motivated by this fact, a new distribution is introduced by adding two extra parameters to a Weibull distribution, which is twice compounding with geometric and zero-truncated Poisson distributions. The new distribution can allow various hazard rate curves that compete well with other alternatives in fitting real data. We derive formal expressions for some of its reliability functions. The maximum likelihood estimation technique is used to estimate the model parameters and a simulation study is conducted to investigate the performance of the maximum likelihood estimates. Finally, an application of the model with a real dataset is presented to illustrate the usefulness of the proposed distribution.

Keywords: Compound distribution, Geometric distribution, Maximum likelihood estimation, Poisson distribution, Weibull distribution.

1 Introduction

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Studying the lifetime of organisms, devices, structures, and materials is of major importance in the biological and engineering sciences. On the one hand, a substantial part of such study is devoted to the mathematical description of the length of life by a failure distribution. On the other hand, the Weibull distribution is one of the most popular models for failure times. In recent years, many authors have proposed generalizations of the Weibull distribution.

Marshall and Olkin [9] proposed adding a parameter to the lifetime distribution through compounding with the geometric distribution. Their suggested method was motivated by a parallel or series system with a random number of components. Their work was extended by Adamidis and Loukas [1] when the two-parameter exponential-geometric (EG) distribution was introduced.

According to Ross [11], any system can be represented either as a series arrangement of parallel structures or as a parallel arrangement of series structures. The proposed family is motivated by a system consisting of parallel components, with each component consists of a series of components, i.e. a system made of parallel and series structures. The purpose of this paper is to introduce a new lifetime distribution by compounding a Weibull, geometric, and zero-truncated Poisson distributions, which is referred to as the Weibull geometric Poisson (WGP) distribution. The compounding procedure follows ideas of Marshall and Olkin [9].Applications of the parallel and series systems can be found in the areas of nuclear power systems (Pham [10]) and modeling crystal deformation (Eichhorn et al. [5]).

The rest of this paper is organized as follows: In Section 2, a new distribution is obtained by mixing the Weibull distribution and the geometric and zero-truncated Poisson distributions. In Section 3, some useful properties of the introduced distribution are discussed. In Section 4, the estimation of parameters is studied by the maximum likelihood method. A simulation study is also performed to assess the performance of the maximum likelihood estimators. An illustrative example based on a real dataset is provided in Section 5.

2 The new distribution

Suppose that a system is made of U parallel components and that the ith component is made of Z_i components working in series. See Figure 1 for an illustration. Let $X_{i,j}$ denote the lifetime of the jth component in the ith series component. Then, the lifetime of the system is

$$X = \max\left\{\min\left\{X_{i,j}\right\}_{j=1}^{Z_i}\right\}_{i=1}^{U}.$$

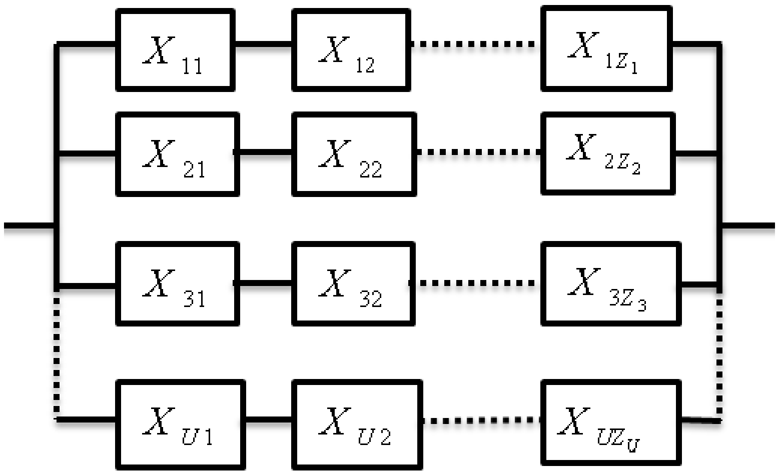

Figure 1: The system is made up of parallel and series components.

Let $X_{i,j}$ are independent and identical, Weibull random variables with the shape parameter α and the scale parameter β has the following probability density function (pdf):

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}};\ x>0,\ \alpha,\beta>0.$$

Let $Z_1, Z_2, \ldots, Z_U$ be independent and identically distributed geometric random variables with parameter θ and the probability mass function (pmf)

$$P(z;\theta) = (1-\theta)\theta^{z-1},\ z = 1,2,\ldots,$$

where $0 < \theta < 1$. Let U be a zero-truncated Poisson random variable with parameter λ and probability mass function

$$P(u;\lambda)=\frac{e^{-\lambda}\lambda^{u}}{(1-e^{-\lambda})u!},\ u=1,2,\ldots,$$

where $\lambda>0$. Assume $X_{i,j}$ and $Z_1,Z_2,\ldots,Z_U$ are independent random variables.

It could be shown that the marginal cumulative distribution function (cdf) of X is

$$\begin{aligned}
F(x;\boldsymbol{\xi}) &= \Pr(X\le x)\\
&= \Pr\left(\max\left\{\min\left\{X_{i,j}\right\}_{j=1}^{Z_i}\right\}_{i=1}^{U}\le x\right)\\
&= \sum_{u=1}^{\infty}\Pr\left(\min\left\{X_{1,j}\right\}_{j=1}^{Z_1}\le x,\ldots,\min\left\{X_{u,j}\right\}_{j=1}^{Z_u}\le x\right)\frac{e^{-\lambda}\lambda^{u}}{(1-e^{-\lambda})u!}\\
&= \sum_{u=1}^{\infty}\left[\Pr\left(\min\left\{X_{1,j}\right\}_{j=1}^{Z_1}\le x\right)\right]^{u}\frac{e^{-\lambda}\lambda^{u}}{(1-e^{-\lambda})u!}\\
&= \frac{e^{-\lambda}}{(1-e^{-\lambda})}\sum_{u=1}^{\infty}\left[1-\Pr\left(\min\left\{X_{1,j}\right\}_{j=1}^{Z_1}\ge x\right)\right]^{u}\frac{\lambda^{u}}{u!}\\
&= \frac{e^{-\lambda}}{(1-e^{-\lambda})}\sum_{u=1}^{\infty}\frac{\lambda^{u}}{u!}\left[1-\Pr\left(X_{1,1}\ge x,\ldots,X_{1,Z_1}\ge x\right)\right]^{u}\\
&= \frac{e^{-\lambda}}{(1-e^{-\lambda})}\sum_{u=1}^{\infty}\frac{\lambda^{u}}{u!}\left\{1-\sum_{z=1}^{\infty}\left[e^{-\beta x^{\alpha}}\right]^{z}(1-\theta)\theta^{z-1}\right\}^{u}\\
&= \frac{e^{-\lambda}}{(1-e^{-\lambda})}\sum_{u=1}^{\infty}\frac{\lambda^{u}}{u!}\left\{1-\frac{1-\theta}{\theta}\sum_{z=1}^{\infty}\left[\theta e^{-\beta x^{\alpha}}\right]^{z}\right\}^{u}\\
&= \frac{e^{-\lambda}}{(1-e^{-\lambda})}\sum_{u=1}^{\infty}\frac{\lambda^{u}}{u!}\left\{\frac{1-e^{-\beta x^{\alpha}}}{1-\theta e^{-\beta x^{\alpha}}}\right\}^{u}\\
&= \left(e^{\lambda}-1\right)^{-1}\left(\exp\left\{\lambda\frac{1-e^{-\beta x^{\alpha}}}{1-\theta e^{-\beta x^{\alpha}}}\right\}-1\right), \qquad (1)
\end{aligned}$$

for $x>0$ where $\boldsymbol{\xi}=(\alpha,\beta,\theta,\lambda)$. The $F(x)$ function in (1) is the cdf of a new distribution here named WGP. The WG distribution of Barreto-Souza et al. [3] is a particular case of the WGP distribution, obtained when $N=1$ is degenerate. The CWP distribution of Mahmoudi and Sepahdar [8] is the particular case of the WGP distribution, obtained when $U = 1$.

3 Some useful properties

Let X be a WGP random variable with cdf (1). The corresponding pdf, survival function, hazard rate (hrf) function, and quantile function are

$$f(x;\boldsymbol{\xi}) = \frac{\alpha\beta\lambda(1-\theta)x^{\alpha-1}e^{-\beta x^{\alpha}}}{\left(1-e^{-\lambda}\right)\left\{1-\theta e^{-\beta x^{\alpha}}\right\}^{2}} \exp\left\{-\frac{\lambda(1-\theta)e^{-\beta x^{\alpha}}}{1-\theta e^{-\beta x^{\alpha}}}\right\},$$

$$S(x;\boldsymbol{\xi}) = \left(1-e^{-\lambda}\right)^{-1}\left(1-\exp\left\{-\frac{\lambda(1-\theta)e^{-\beta x^{\alpha}}}{1-\theta e^{-\beta x^{\alpha}}}\right\}\right), \tag{2}$$

$$h(x;\boldsymbol{\xi}) = \frac{\alpha\beta\lambda(1-\theta)x^{\alpha-1}e^{-\beta x^{\alpha}}}{\left\{1-\theta e^{-\beta x^{\alpha}}\right\}^{2}}\left(\exp\left\{\frac{\lambda(1-\theta)e^{-\beta x^{\alpha}}}{1-\theta e^{-\beta x^{\alpha}}}\right\}-1\right)^{-1},$$

and

$$Q(u;\boldsymbol{\xi}) = -\frac{1}{\beta}\log\left\{\frac{1-\frac{1}{\lambda}\log\left(1+u\left[e^{\lambda}-1\right]\right)}{1-\frac{\theta}{\lambda}\log\left(1+u\left[e^{\lambda}-1\right]\right)}\right\}, \tag{3}$$

respectively. Figure 2 displays the density and hazard rate functions of the WGP distribution for some selected parameter values.
In particular,

$$\text{Median}(\text{X}) - \frac{1}{\beta}\log\left\{\frac{1-\frac{1}{\lambda}\log\left(1+0.5\left[e^{\lambda}-1\right]\right)}{1-\frac{\theta}{\lambda}\log\left(1+0.5\left[e^{\lambda}-1\right]\right)}\right\}.$$

Finally, If U is a uniform$[0,1]$ random variable, then $Q(U)$ is a WGP random variable.

Let X be a WGP random variable with cdf (1). Using the concept of power series and some other mathematical expansions, we derived a linear representation for the survival function of the WPG distribution. Since

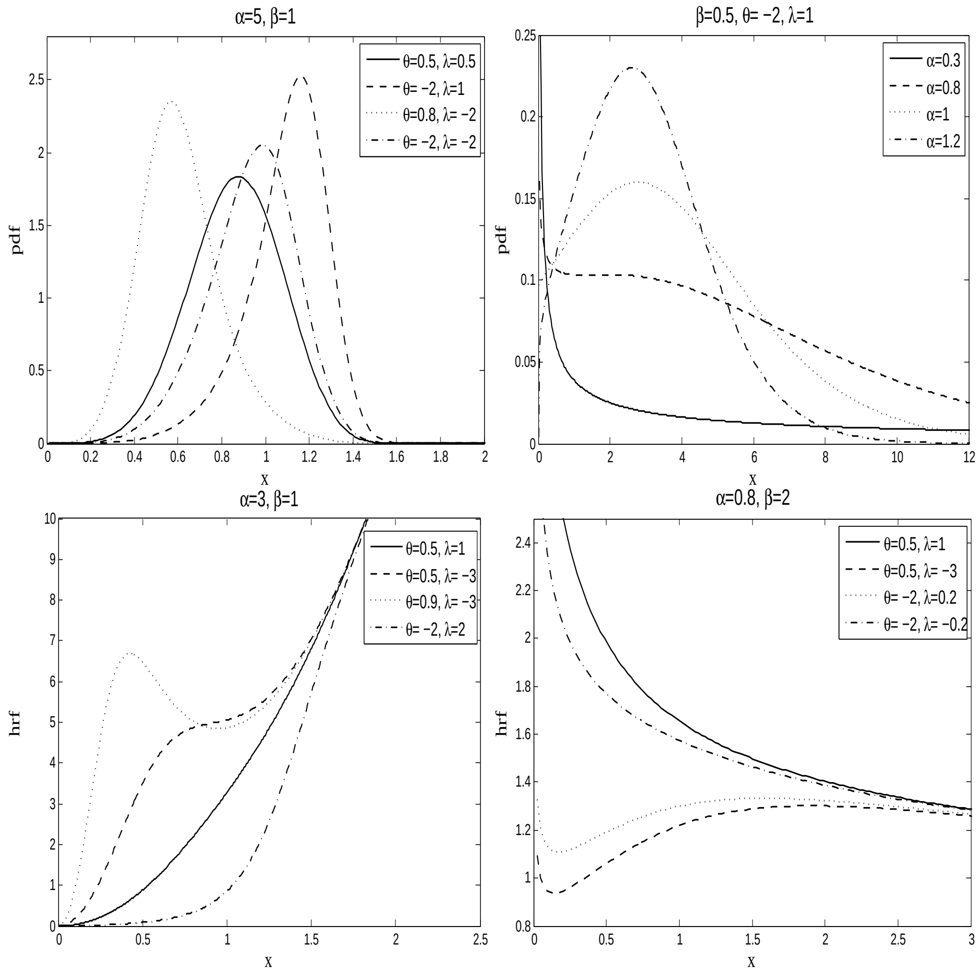

Figure 2: Graphs of the pdf (first row) and hrf (second row) of the WGP distribution for selected values of the parameters.

$$\left(\frac{y}{1-\theta y}\right)^{j} = \sum_{k=0}^{\infty} (-1)^{k} \binom{-j}{k} \theta^{k} y^{j+k},$$

then equation (2) can be expressed as

$$S(x;\boldsymbol{\xi}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{j,k}(\theta,\lambda) e^{-(j+k)\beta x^{\alpha}}, \tag{4}$$

where

$$c_{j,k}(\theta,\lambda) = \binom{-j}{k} \frac{(-1)^{k+j-1}\lambda^{j}\theta^{k}(1-\theta)^{j}}{j!}$$

Equation (4) is the main result of this section. So, several mathematical properties of the proposed family such as moments and moment generating function can be obtained by using this expansion. The formula for the rth moment of X is obtained from (4) as

$$E\left[X^r\right] = r\int_0^\infty x^{r-1}S(x)dx = \frac{r\Gamma\left(\frac{r}{\alpha}\right)}{\alpha\beta^{\frac{r}{\alpha}}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{c_{j,k}(\theta,\lambda)}{(j+k)^{\frac{r}{\alpha}}}.$$

4 Estimation and Simulation

In this section, we discuss the estimation of the parameters of the WGP distribution. Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random sample from the WGP distribution with observed values $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and parameters $\boldsymbol{\xi} = (\alpha, \beta, \theta, \lambda)$. The log-likelihood function based on the observed sample is

$$\begin{aligned}\ell(\boldsymbol{\xi} \mid \mathbf{x}) &= n\log\left[\frac{\alpha\beta\lambda(1-\theta)}{1-e^{-\lambda}}\right] + (\alpha-1)\sum_{i=1}^{n}\log\left[x_i\right] - \beta\sum_{i=1}^{n}x_i^{\alpha} \\ &\quad -2\sum_{i=1}^{n}\log\left[1-\theta e^{-\beta x_i^{\alpha}}\right] - \lambda(1-\theta)\sum_{i=1}^{n}\frac{e^{-\beta x_i^{\alpha}}}{1-\theta e^{-\beta x_i^{\alpha}}}. \qquad (5)\end{aligned}$$

The maximum likelihood estimate (MLE) of $\boldsymbol{\xi}$ called $\widehat{\boldsymbol{\xi}}$ should satisfy the following equation $U_n(\boldsymbol{\xi}) = (\partial\ell/\partial\alpha, \partial\ell/\partial\beta, \partial\ell/\partial\theta, \partial\ell/\partial\lambda) = \mathbf{0}$. The solution of this nonlinear system of equations has no closed-form. To solve this equation, it is usually more convenient to use nonlinear optimization algorithms such as the quasi-Newton algorithm to maximize the log-likelihood function numerically. In the application section, the MLEs were obtained by directly maximizing (5), concerning to the parameters. The optim routine in R was used for maximization.

We also conducted a simulation study to assess the performance of the maximum likelihood estimation procedure for estimating the WGP distribution parameters using (3). Samples of sizes 10, 12, 14, ..., 200 are generated for parameter vector $\boldsymbol{\xi} = (1, 2, 0.5, 0.9)$ from WGP distribution.

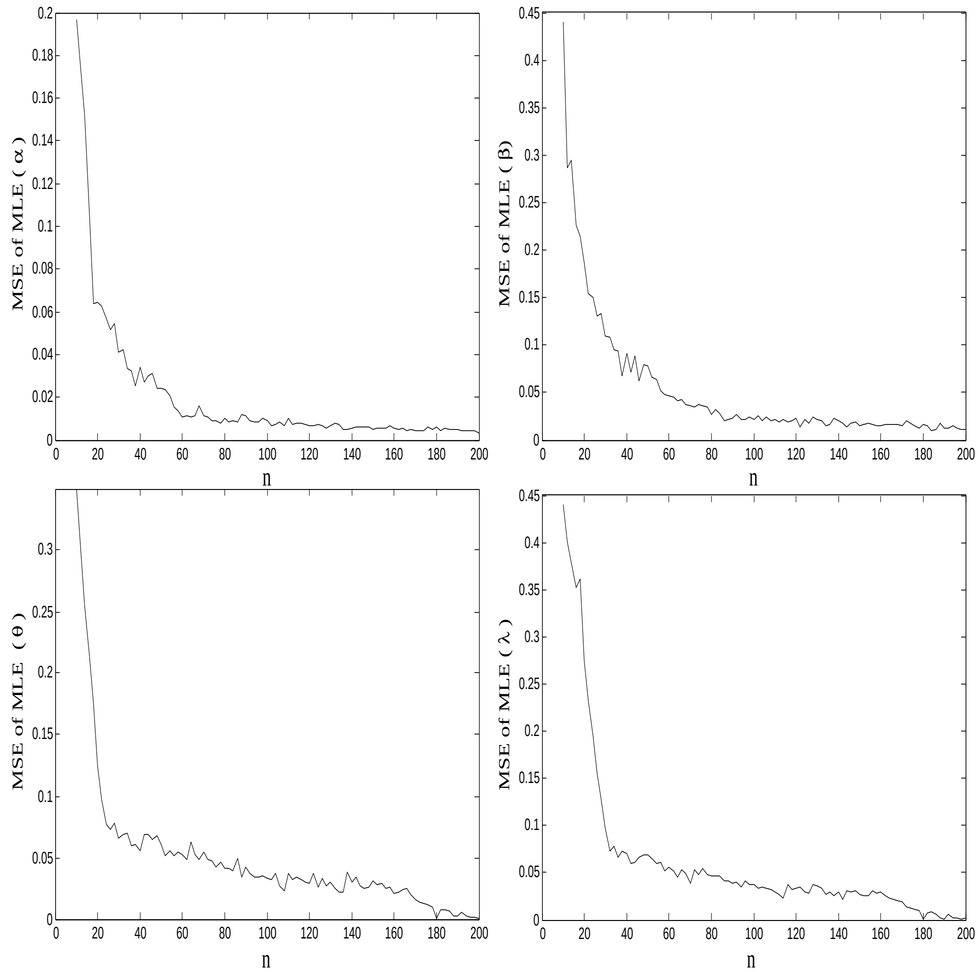

Figure 3: From top to bottom and from left to right: $\mathrm{MSE}_{\alpha}(n), \mathrm{MSE}_{\beta}(n), \mathrm{MSE}_{\theta}(n), \mathrm{MSE}_{\lambda}(n)$.

We repeated the simulation $k = 1000$ times and calculated the MLEs and the bias and mean squared error (MSE) of the parameter estimates. The empirical results are given in Figure 3 indicates that the maximum likelihood estimators carry out well for estimating the parameters of the WGP model. According to Figure 3, it can be concluded that as the sample size n increases, the MSEs decay toward zero. We have presented results for only one choice for $\boldsymbol{\xi} = (1, 2, 0.5, 0.9)$. However, the results were similar for a wide range of other choices.

5 Data Application

In this section, we fit the WGP distribution to a real dataset using maximum likelihood method. First, we give the MLEs and the corresponding standard errors of the model parameters and the values of the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICc), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov (K-S) statistics. The lower the values of these criteria, the better the fit. Finally, we provide the histogram of estimated pdfs for the dataset to visualize the fitted models.

The dataset consists of the strength of 1.5 cm glass fibers, measured at the National physical laboratory, England (see Smith and Naylor [12]). Table 1 gives some descriptive statistics for this dataset.

Table 1: Descriptive statistics for the glass fibers dataset.

n	Mean	Q_1	Median	Q_3	Mode	Variance	Skewness	Kurtosis	Min	Max
63	1.507	1.375	1.59	1.685	1.61	0.105	-0.90	3.92	0.55	2.24

Larger parameter space of the distribution provides a better fit. It can be expected to provide. So, we extended the parameter space of θ to $(-\infty, 1)$, the stated pdfs remain valid pdfs over the extended space. We also extended the parameter space of λ for the zero-truncated Poisson distribution to $(-\infty, \infty)$, the WGP pdf remains a valid pdf over the extended space. For more details, see Goldoust et al. [7].

The MLEs of the parameters are computed and the goodness-of-fit statistics for these models are compared with fit of the popular exponential, Weibull, Odd Weibull (Cooray [4]), beta Weibull (BW) (Famoye et al. [6]), and beta generalized exponential (BGE) (Barreto-Souza et al. [2]) distributions. The MLEs, log-likelihood value, the corresponding standard errors, the Kolmogorov-Smirnov statistic, AIC, AICc, and BIC values are shown in Table 2. We can see that the largest log-likelihood value, the smallest AIC, AICc, and BIC values are obtained for the WGP model and it shows that the WGP distribution gives the best fit with respect to all indices.

Table 2: Estimates and goodness-of-fit measures for the glass fibers dataset.

Model	Estimated parameters	$\log(\ell)$	K-S	AIC	AICc	BIC
Exponential	0.6636	-88.83	0.632	179.26	179.73	181.80
SE	(0.0836)					
Weibull	5.7807, 0.0597	-15.21	0.144	34.41	34.57	38.70
SE	(0.9532, 0.3284)					
OW	6.0258, 0.0539, 0.9438	-15.187	0.155	36.374	37.064	42.803
SE	(1.3333, 0.0331, 0.2667)					
BW	7.0138, 0.5533, 0.4498, 0.0499	-13.044	0.118	34.088	35.141	42.661
SE	(0.8896, 0.6459, 0.1810, 0.0464)					
BGE	22.6124, 0.9227, 0.4125, 93.4655	-15.599	0.158	39.198	40.251	47.771
SE	(22.8153, 0.5135, 0.3152, 116.6665)					
WGP	3.2061, 0.6888, -11.4004, 0.5528	-12.029	0.105	30.058	30.465	36.487
SE	(0.9476, 0.5627, 27.2243, 3.2966)					

The density graphs for the fit of the distributions for the glass fibers dataset are shown in Figure 4. The fitted pdf of the WGP distribution captures the observed histograms better than others. Hence, we can say that the WGP distribution provides the best fit for at least a real dataset.

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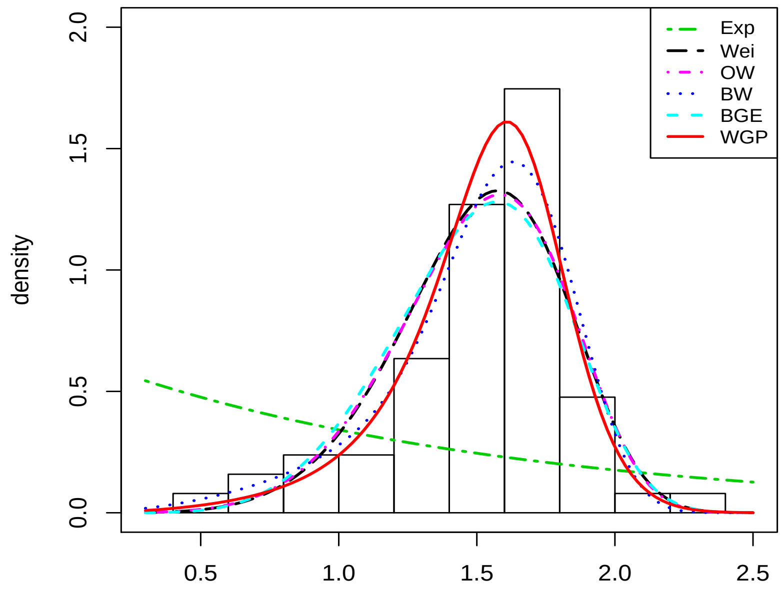

Figure 4: Graph of the estimated pdfs of the WGP and other competitive models for the glass fibers dataset.

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Optimal Design of a Simple Step-stress Accelerated Life Test with Interval Monitoring and Progressive Type-I Censoring

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Abstract: In order to gather the information about the lifetime distribution of a product, a standard life testing method at normal operating conditions is not practical when the product has an extremely long lifespan. Accelerated life testing solves this difficult issue by subjecting the test units at higher stress levels than normal for quicker and more failure data. The lifetime at the design stress is then estimated through extrapolation using an appropriate regression model. Although continuous monitoring of the exact failure times is an ideal mode, the exact failure times of test units may not be available in practice due to technical limitations and/or budgetary constraints, but only the failure counts are collected at certain time points during the test (i.e., interval monitoring). In this work, the optimal design of a simple step-stress accelerated life test with interval monitoring under progressive Type-I censoring is studied for assessing the reliability characteristics. The nature of the optimal stress duration is demonstrated under various design criteria. These optimal designs are investigated in detail for exponential lifetimes with a single stress variable, and the effect of the intermediate censoring proportion on the optimal design is presented.

Keywords: Accelerated life tests, Design of experiment, Interval monitoring,

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Progressive Type-I censoring, Step-stress loading.

1 Introduction

With ever increasing reliability and substantially long life-spans of products, it is often very difficult for standard life testing methods under normal operating conditions to obtain sufficient information about the failure time distribution of the products. This practical difficulty is overcome by accelerated life tests (ALT). By subjecting test units to higher stress levels than normal, the ALT collects more failure data in a shorter period of time. By applying more severe stresses, ALT collects information on the parameters of lifetime distributions more quickly. The lifetime at the normal operating stress can be estimated through extrapolation using an appropriate stress-response regression model. As a special mode of ALT, a simple (step-up) step-stress test implements an increase of the stress level at a prefixed time point Δ during the test until the termination time τ. In the past decades, the inference and design optimization for the step-stress ALT have attracted great attention in the reliability and engineering literature; see [3], [2] for example.

Furthermore, due to time and resource constraints, censored sampling is usually necessary in practice, and in particular, a generalized censoring scheme known as progressive Type-I censoring allows functional test units to be withdrawn successively from the experiment at some prefixed non-terminal time points. Those withdrawn unfailed units can be used in other tests in the same or at a different facility. This work formulates the optimization of a simple step-stress ALT under progressive Type-I censoring. It is assumed here that the lifetimes of the test units are from an exponential distribution at each stress level, and a log-linear relationship is held between the stress level and the mean lifetime parameter. To explain the effect of increasing stress, the accelerated failure time (AFT) model is also adopted. Using three different design criteria, including D-optimality, C-optimality and A-optimality, the optimal design is investigated in detail with a single stress variable, and the effect of the intermediate censoring proportion on the optimal design is numerically studied.

Due to technical limitations and budgetary constraints, it is also considered that the exact failure times of test units may not be observable (i.e., interval monitoring). The interval monitoring occurs when only the failure counts are collected at certain time points during the test. Although continuous monitoring of the exact failure times is ideal, the exact failure times of test units may not be available in practice due to technical limitations and/or budgetary constraints.

The rest of the article is structured as follows. Section 2 presents the model description for a simple step-stress ALT under progressive Type-I censoring. The (expected) Fisher information matrix of the regression parameters is derived in Section 3. Based on the Fisher information obtained in Section 4, various optimality criteria are defined. Section 5 provides the results of a numerical study to investigate the behaviors of the optimal step durations. Section 6 is devoted to the concluding remarks.

2 Model descriptions and MLE

Let $s(t)$ be the given stress loading (a deterministic function of time) for ALT. Also, let s_H be an upper bound of stress level and s_U be the normal use-stress level. The standardized stress loading is then defined as

$$x(t) = \frac{s(t) - s_U}{s_H - s_U}, \quad t \geq 0$$

so that the range of $x(t)$ is $[0,1]$. Now, let us define $0 = x_0 < x_1 < x_2 \leq 1$ to be the ordered standardized stress levels to be used in the test. It is further assumed that under any stress level x_i, the lifetime of a test unit follows an exponential distribution whose probability density function (PDF) and cumulative distribution function (CDF) are

$$f_i(t) = \frac{1}{\theta_i} e^{-\frac{t}{\theta_i}}, \quad 0 < t < \infty, \tag{1}$$

$$F_i(t) = 1 - e^{-\frac{t}{\theta_i}}, \quad 0 < t < \infty, \tag{2}$$

respectively. Also, it is assumed that under any stress level x_i, the mean time to failure (MTTF) of a test unit, θ_i, is a log-linear function of stress given by

$$\log(\theta_i) = \alpha + \beta x_i, \tag{3}$$

where the regression parameters α and β need to be estimated. The log-linear relationship is a commonly used and well-studied model for the accelerated exponential distribution model. Along with its simplicity, the log-linear link represents several significant life-stress relationships built from physical principles such as Arrhenius, inverse power law, Eyring, temperature-humidity, and temperature-non-thermal; see Miller and Nelson [3].

Total $N_1 = n$ test units are initially placed at stress level x_1 and tested until time Δ at which point c_1 live items are arbitrarily withdrawn from the test and the stress is changed to x_2. The test is then continued on $N_2 = n - n_1 - c_1$ surviving units until time τ, at which all the $c_2 = N_2 - n_2 = n - n_1 - n_2 - c_1$ remaining items are withdrawn, thereby terminating the test. Now, let n_1 denote the number of units failed at stress level x_1 in time interval $[0, \Delta)$ while n_2 denotes the number of units failed at x_2 in $[\Delta, \tau)$. For the continuous monitoring, we also have failure times of failed units. It is noted that these exact failure times are not available under the interval monitoring. Furthermore, let c_1 be the number of units censored at time Δ. Under this setup, a simple step-stress ALT under progressive Type-I censoring proceeds as follows.

Since the stress-loading is non-constant for the step-stress ALT, an additional model to explain the effect of changing stress is required. In reliability engineering, the AFT model, also referred to as the additive accumulative damage model, is often appropriate as it generalizes several well-known models for the exponential distribution, including the basic (linear) cumulative exposure model. Under the AFT model along with (1) and (9), the PDF and CDF of a test unit for the simple step-stress ALT are

$$f(t) = \begin{cases} f_1(t), & 0 \le t < \Delta, \\ S_1(\Delta) f_2(t - \Delta), & \Delta \le t < \tau. \end{cases} \tag{4}$$

$$F(t)=\begin{cases}1-S_1(t), & 0\le t<\Delta,\\ 1-S_1(\Delta)S_2(t-\Delta), & \Delta\le t<\tau.\end{cases} \tag{5}$$

respectively. Now, based on (4) and (5), the likelihood and MLE of the regression parameters α and β are derived in the following.

The joint probability mass function (JPMF) of $\mathbf{n}=(n_1,n_2)$ is obtained by using (5) as

$$f_J(\mathbf{n})=\binom{n}{n_1}\binom{N_2}{n_2}\Big[\prod_{i=1}^{2}(1-\exp(\frac{\Delta_i}{\theta_i}))^{n_i}\Big]\exp(-\sum_{i=1}^{2}\frac{\Delta_i}{\theta_i}(N_i-n_i)) \tag{6}$$

where $\Delta_1=\Delta$ and $\Delta_2=\tau-\Delta$. With (6) and the log-linear link given in (3), the log-likelihood function of (α,β) is written as

$$\ell=\sum_{i=1}^{2}n_i\log\left(1-e^{-\Delta_i\exp[-(\alpha+\beta x_i)]}\right)-\sum_{i=1}^{2}\Delta_i(N_i-n_i)\exp[-(\alpha+\beta x_i)]. \tag{7}$$

Upon differentiating (7) with respect to α and β, the MLE $\hat{\alpha}$ and $\hat{\beta}$ can be obtained as simultaneous solutions to the following two equations:

$$\sum_{i=1}^{2}n_i\Delta_i\frac{\exp(-(\alpha+\beta x_i)-\Delta_i e^{-(\alpha+\beta x_i)})}{1-e^{-\Delta_i\exp[-(\alpha+\beta x_i)]}}=\sum_{i=1}^{2}\Delta_i(N_i-n_i)\exp[-(\alpha+\beta x_i)],$$

$$\sum_{i=1}^{2}n_i x_i\Delta_i\frac{\exp(-(\alpha+\beta x_i)-\Delta_i e^{-(\alpha+\beta x_i)})}{1-e^{-\Delta_i\exp[-(\alpha+\beta x_i)]}}=\sum_{i=1}^{2}x_i\Delta_i(N_i-n_i)\exp[-(\alpha+\beta x_i)].$$

The MLE $\hat{\alpha}$ and $\hat{\beta}$ do not exhibit explicit forms, and solving this series of equations requires a computational technique. Also, statistical inferences with these MLE are based on the asymptotic result that $(\hat{\alpha},\hat{\beta})$ follows an approximate bivariate normal distribution with mean (α,β) and variance-covariance matrix $\mathbf{I}^{-1}(\alpha,\beta)$.

3 Censoring scheme and Fisher information matrix

Prefixing c_1 bears an intrinsic mathematical lapse for a simple step-stress ALT under progressive Type-I censoring since that it is possible that all the units fail before reaching the stress level x_2, resulting in an early termination of the

ALT and failing to realize c_1. For this reason, Gouno et al. [1], for example, assumed a large sample size, small global censoring proportions, and a small number of stress levels for an approximate/asymptotic analysis of progressively Type-I censored data so that the prefixed number of units could be removed at the end of each stage. A usual life test, however, runs on a small sample size and there might be severe censoring due to budgetary constraints and/or facility requirements. In such situations, the assumption of a large sample is violated and consequently, the progressive censoring scheme needs to be modified to ensure its feasibility. An easy practical modification for a simple step-stress ALT is to decide on the fixed proportion $0 \leq \pi_1^* < 1$ of surviving items to be censored at Δ. Since all the remaining units withdraw from the test at τ, one could define $\pi_2^* = 1$. Then, the actual number of items censored at Δ is determined by $c_1 = (n-n_1)\pi_1^*$, where π_1^* is the fixed intermediate censoring proportion. This modification mathematically allows the ALT to not terminate before reaching the stress level x_2. Since the number of surviving units at Δ before censoring takes place is random, c_1 is essentially a random quantity under the suggested censoring mode.

Then, based on the log-likelihoods and (7) obtained in the preceding section, the expected Fisher information matrix $\mathbf{I}(\alpha,\beta)$ is derived in a common form as

$$\mathbf{I}(\alpha,\beta) = n \begin{bmatrix} \sum_{i=1}^{2} A_i & \sum_{i=1}^{2} A_i x_i \\ \sum_{i=1}^{2} A_i x_i & \sum_{i=1}^{2} A_i x_i^2 \end{bmatrix}, \tag{8}$$

where

$$\begin{aligned} A_1 &= \frac{\Delta^2}{\theta_1^2}\frac{S_1(\Delta)}{F_1(\Delta)}, \\ A_2 &= \frac{(\tau-\Delta)^2}{\theta_2^2}\frac{S_2(\tau-\Delta)}{F_2(\tau-\Delta)}S_1(\Delta)(1-\pi_1^*). \end{aligned} \tag{9}$$

By utilizing the distributional properties:

- n_1 has a binomial distribution with parameters n and $F_1(\Delta_1)$;

- n_2 given N_2 has a binomial distribution with parameters N_2 and $F_2(\Delta_2) = \frac{F(\tau)-F(\Delta)}{1-F(\Delta)}$.

4 Design criteria and optimal step duration

A number of design criteria were considered in this study for determining the optimal step duration Δ^*. These objective functions are based on the Fisher information matrix $\mathbf{I}(\alpha,\beta)$ derived in the preceding section.

4.1 D-optimality

Design optimality criterion often used in planning ALT is based on the reciprocal of the determinant of the Fisher information matrix, or equivalently, the determinant of the asymptotic variance-covariance matrix. The overall volume of the Wald-type joint confidence region of (α,β) is proportional to $|\mathbf{I}^{-1}(\alpha,\beta)|^{1/2}$ at a fixed level of confidence. In other words, it is inversely proportional to $|\mathbf{I}(\alpha,\beta)|^{1/2}$, the square root of the determinant of $\mathbf{I}(\alpha,\beta)$. Accordingly, a larger value of $|\mathbf{I}(\alpha,\beta)|$ would correspond to a smaller asymptotic joint confidence ellipsoid of (α,β) and thus a higher joint precision of the estimators of α and β. Based on this, an objective function is defined as

$$\phi_D(\Delta) = n^2\,|\mathbf{I}^{-1}(\alpha,\beta)|, \tag{10}$$

and the D-optimal step duration Δ^*_D is obtained by minimizing (10) for the maximal joint precision of $(\hat{\alpha},\hat{\beta})$.

4.2 C-optimality

ALT often aims to estimate the parameters of interest with maximum precision and minimum variability possible. For the step-stress ALT, such a parameter of interest is θ_0, the MTTF at the normal operating stress level x_0. Based on (8), an objective function to serve this purpose is defined as

$$\phi_C(\Delta) = nAVar(\log\hat{\theta}_0) = nAVar(\hat{\alpha}+\hat{\beta}x_0) = n(1\ 0)\mathbf{I}^{-1}(\alpha,\beta)(1\ 0)'. \quad (11)$$

The C-optimal step duration Δ_C^* is the one that minimizes the objective function in (11).

4.3 A-optimality

Another design optimality criterion considered in this study is based on the trace of the first-order approximation of the variance-covariance matrix of the MLE, or the sum of the diagonal elements of $\mathbf{I}^{-1}(\alpha,\beta)$. This A-optimality criterion provides an overall measure of the average variance of the parameter estimates and gives the sum of the eigenvalues of the inverse of the Fisher information matrix. The A-optimal step duration Δ_A^* minimizes the objective function defined as

$$\phi_A(\Delta) = ntr(\mathbf{I}^{-1}(\alpha,\beta)). \quad (12)$$

5 Comparative numerical study

A computational study was conducted to investigate the optimal step durations for a progressively Type-I censored simple step-stress ALT under various design criteria discussed in Section 4, and the results are presented in this section. The behaviors of these optimal stress change points were also evaluated as a function of varying parameters such as the sample size, total test duration, MTTF, and the degree of censoring. For an illustrative purpose, standardized equi-spaced stress levels $x_i = x_0 + id$ were considered with the use-stress level $x_0 = 0$ and the stress increment $d = 0.25$.

Upon this layout, the results of the D-optimality and C-optimality criteria are independent of the choice of d while A-optimality criterion is dependent on it in varying degrees. An extensive and comprehensive numerical study was conducted with exhaustive combinations of various parameter values. In contrast to the work of [1], which determined the optimal step-stress ALT with uniform step durations, our design optimization is performed with respect to flexible, non-uniform durations, for which the total test duration τ is a critical parameter to define a valid search region. Thus, rather than tabulating specific values, the computational results provided here are intended to visualize information about the way the optimal step duration to the total test duration Δ^*/τ changes as a function of the test duration τ and other relevant parameters.

Figure 1 presents the optimal step durations Δ^* with respect to the total test durations τ under each design criterion with $\theta_0 = 1000$, $\theta_1 = 300$, $\theta_2 = 90$ and 10% intermediate censoring over the remaining test units at the end of the first stress level. It is observed that Δ^* with respect to τ exhibits similar behaviors across different design criteria. The ratio Δ^*/τ is convex in τ, meaning that there exists the minimal ratio with respect to τ and a larger proportion of the test duration is assigned to the first stress level as τ increases. It is interesting to note that the results of the A-optimality and the C-optimality are very similar in terms of the optimal step duration and the corresponding optimum. When the total test duration τ is very short (i:e:, $\tau \to 0$), it is observed that, the C-optimal duration of the first stress level is the largest, followed by the A-optimal duration, then by the D-optimal duration. In particular, the D-optimal proportion Δ^*/τ always starts from 50%.

As shown in Figure 2, for higher intermediate censoring proportion π_1^*, the behaviors of the optimal step durations Δ^* with respect to the total test duration τ do not seem affected much.

Figures 3 and 4 present the corresponding values of the optima $\phi(\Delta^*)$ of each objective function, achieved by the optimal step durations Δ^* presented in Figures 1 and 2. These optima are plotted as a function of the total test duration τ. The optima of each design criterion decrease at first and then increase as τ gets longer. This convexity indicates that if the total test duration is longer than necessary, it actually hurts the design, producing a sub-optimal step-stress ALT. It means that the total test duration τ is an important dimension to consider when constructing the optimal step-stress ALT under the interval monitoring.

6 Conclusion

This work investigates the design optimization of a simple step-stress ALT under progressive Type-I censoring with non-uniform step durations. The interval monitoring was considered. Using a log-linear relationship between the MTTF parameter and the (transformed) stress level along with the AFT model for the effect of increasing stress levels, the Fisher information was derived for the model parameters. Based on the Fisher information matrix, the objective functions were then defined under several design criteria, including the D-optimality, C-optimality and A-optimality. For exponential failure times, the effects of the intermediate censoring proportion π_1^* and the test duration τ on the relative design efficiencies were also explored.

In this work, the optimal designs were formulated by implementing unequal or flexible durations at different stress levels. It is of practical interest to examine if the optimal designs under time constraints utilize all the stress levels when more than two levels are available. Our preliminary analysis reveals that the optimal step-stress ALT may implement more than two stress levels depending on the total test duration τ.

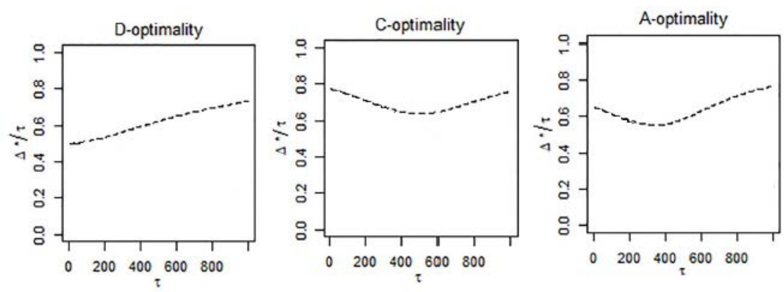

Figure 1: Optimal step durations Δ^* with respect to the total test durations τ for progressively Type-I censored simple step-stress ALT with $\pi_1^* = 10\%$.

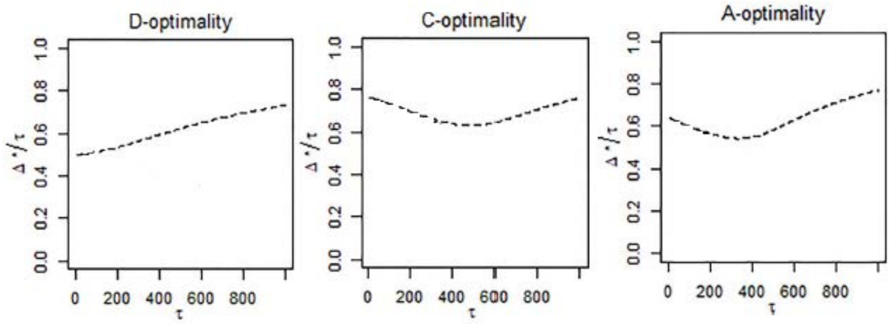

Figure 2: Optimal step durations Δ^* with respect to the total test durations τ for progressively Type-I censored simple step-stress ALT with $\pi_1^* = 20\%$.

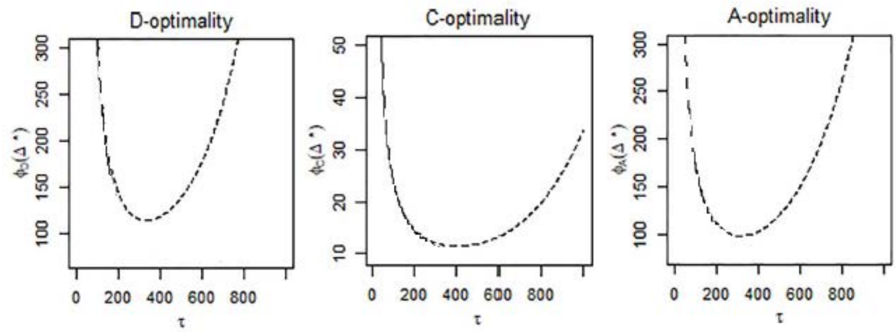

Figure 3: Corresponding optima $\phi(\Delta^*)$ of the objective functions with respect to the total test durations τ for progressively Type-I censored simple step-stress ALT with $\pi_1^* = 10\%$.

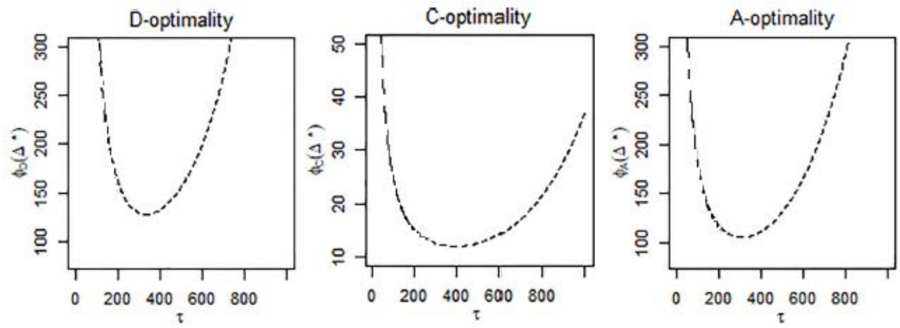

Figure 4: Corresponding optima $\phi(\Delta^*)$ of the objective functions with respect to the total test durations τ for progressively Type-I censored simple step-stress ALT with $\pi_1^* = 20\%$.

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Reliability Evaluation of Weighted Systems Consisting of Multiple Types of Components

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Abstract: In the present paper, a weighted k-out-of n system with $M \geq 2$ different types of components is considered. The extended survival signature for such systems is defined according to a model of failure for the system. Reliability function of the system and other reliability indices are obtained based on the proposed extended survival signature. Numerical examples and some applications are posted to illustrate the model.

Keywords: Extended survival signature, Signature vector, Weighted k-out-of-n systems.

1 Introduction

Weighted systems and specially weighted k-out-of-n systems have captured the attention of many researchers in recent years because of their wide range of applications. A system with n components is said to have an ordinary k-out-of-n structure, if it operates as long as at least k components out of the n components operate. In the n-component weighted system, components will

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be supposed to have different loads or capacities, w_i, say, $i = 1,2,...,n$. A weighted k-out-of-n system fails whenever the accumulated weight of the functioning components becomes less than a predetermined threshold k, where $(\min w_i \leq k \leq \sum_{i=1}^{n} w_i)$, $i = 1,2,...,n$. Weighted k-out-of-n systems were introduced for the first time by Wu and Chen[1]. Samaniego and Shaked[7] studied a more general case in which the weights and the threshold of the system were allowed to take on any positive values. Eryilmaz and Sarikaya[5] study reliability analyses for weighted systems with n components classified in two groups according to their weights. Weighted systems have many applications in real-life. For example, oil transmission system consists of several pipelines with different diameters (capacities) could be treated as a weighted system with pipes considered as components. On the other hand, the vast majority of real-life systems contain components not only with different capacities but also from different types. In this case, the system contains n components, each component has its own capacity (weight) and the components are selected from M different types $M \geq 2$. Each type has n_i of components, whereas $\sum_{i=1}^{n} n_i = n$. In the case of $M = 1$ and $n_1 = n$, we get the typical weighted system with identical components. Also, as a special case of $M = n$ and $n_i = 1$, for all $i = 1,2,...,M$, we have a system with nonidentical components. Many examples could be mentioned for such systems. Wind turbine plants with different generating capacity in power generation systems are good examples for weighted k-out-of-n systems, Eryilmaz[9], Louie and Sloughter[8]. When the turbines in the same plant are of the same model, whereas different models selected for different plants, then this system could be classified as a multi-type weighted system. Although there are many examples of such systems in real life, there is a lack of studies of these systems in the literature. For example, Eryilmaz and Sarikaya[5] studied reliability properties of a special case when a weighted system consists of two different types of components with a fixed weight and a common reliability function for all components in each type. Salehi et al [4] studied the reliability evaluation of a system with two types of components in the case that a random number of components from one type are chosen.

In system reliability, the concept of signature vector introduced by Samaneigo[2] is a very useful tool that has been widely used for the reliability evaluation of technical systems. A system with n component has a signature vector of the form $p = (p_1, p_2, ..., p_n)$, where the element p_i refers to the probability that the failure of the ith ordered component leads to the failure of the whole system. If we denote the system lifetime by T_s and the lifetime of components as $X_1, X_2, ..., X_n$, and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ the ordered lifetimes of components, then $p_i = P(T_s = X_{i:n})$. If the components of the system are supposed to beindependent and identically distributed (**iid**) with a common cumulative distribution function F, it will be easy to verify that

$$\overline{F}_s(t) := P(T_s > t) = \sum_{i=1}^{n} p_i P(X_{i:n} \geq t) = \sum_{i=1}^{n} p_i \overline{F}_{i:n}(t).$$

It is obvious from the above expression that the structure of the system is fully taken into account, whereas it is completely free of distribution. Signature vector for the case of weighted systems was also studied in [3]. However, the weakness of the signature vector is the restriction on the components to be from a single type. This limitation makes it inapplicable for most practical systems like systems with multiple types of components. That is why the concept of survival signature introduced to fill this void in such systems. Coolen and Coolen-Maturi [10] introduced this concept to study systems with $M \geq 2$ types of components. Indeed, survival signature for a system with M different types of components is a multivariate function $\phi(l_1, l_2, ..., l_M)$ which denotes the probability that the system is functioning with exactly l_1 components from the first type, l_2 components from the second type, ... and l_M components from the Mth type. In the present paper, we will generalize this concept for the case of weighted k-out-of-n systems with multiple types of components.

The rest of this paper is organized as follows. In Section 2, we introduce the concept of extended survival signature for the weighted systems and evaluating the reliability functions of such systems. Section 3 is devoted to applications and illustrative examples.

2 Extended survival signature and reliability evaluation

Consider a weighted k-out-of n system with n components. Suppose that the components are selected from $2 \leq M \leq n$ different types. Type i contains a certain number of components, n_i, say, where $n = \sum_{i=1}^{M} n_i$. All components in the same type are supposed to be **iid** with a cumulative distribution function $F_i(t)$. Furthermore, completely independence is also supposed between the lifetimes of different types. Let $w_j^{(i)}$ denote the weight of the component j from the ith type of the system, $i = 1,2,...,M$, $j = 1,2,...,n_i$. Therefore the vector w in the form $w = (w_1^{(1)},...,w_{n_1}^{(1)},w_1^{(2)},...,w_{n_2}^{(2)},...,w_1^{(M)},...,w_{n_M}^{(M)})$ could be regarded as the weights vector for the system. Furthermore, at any given time instant t, the vector $\varepsilon(t) = (\varepsilon_1^{(1)}(t),...,\varepsilon_{n_1}^{(1)}(t),\varepsilon_1^{(2)},...,\varepsilon_{n_2}^{(2)}(t),...,\varepsilon_1^{(M)},...,\varepsilon_{n_M}^{(M)}(t))$ will represent the state vector of the system at time t, where $\varepsilon_j^{(i)}(t)$ indicates the status of the component j from type i, $i = 1,2,...,M$, $j = 1,2,...,n_i$ at time t, i.e., $\varepsilon_j^{(i)}(t) = 1$ in the functioning case and $\varepsilon_j^{(i)}(t) = 0$, otherwise. In what follows, we will consider a classic failure model of the multi-type weighted k-out-of-n system, i.e, the system works if the total weight of the working components at time t is greater than or equal to the predetermined threshold k. If $\Phi(\varepsilon(t))$ is the structure function and denotes the system state at t, then

$$\Phi(\varepsilon(t)) = \begin{cases} 1, & \text{if } \sum_{i=1}^{M}\sum_{j=1}^{n_i} w_j^{(i)} I(\varepsilon_j^{(i)}(t) = 1) \geq k \\ 0, & \text{otherwise,} \end{cases}$$

where $I_{()}$ denotes the indicator function. At a fixed time instant t, if we denote the system lifetime by T_s, we are interested in obtaining the reliability function representation of such system $P(T_s > t)$, using the concept of the extended survival signature. Let $\phi(l_1,l_2,...,l_M)$, denote the probability that the system functions given that exactly l_1 components from the first type, exactly l_2 components from the second type, ... and exactly l_M components from the M-th type, where, $0 \leq l_i \leq n_i$. Indeed, this function is called *the extended survival signature* with M variables (dimensions). In our study, we restrict our attention to the class of coherent systems, for which $\phi(l_1, l_2, ..., l_M)$ is a non-decreasing function of l_i, $i = 1, 2, ..., M$ and $\phi(0, 0, ..., 0) = 1 - \phi(1, 1, ..., 1) = 0$. Generally, this function will be computed for $(n_1 + 1) \times (n_2 + 1) \times ... \times (n_M + 1)$ permutations of elements. In other words, there is $\binom{n_i}{l_i}$ number of ways to select l_i functioning components from the type i to set the respective state vector $v^i_{l_i}$, $i = 1, 2, ..., M$, where $v^i_{l_i} = (v_{i,1}, v_{i,2}, ..., v_{i,n_i})$ for which $\sum_{j=1}^{n_i} v_{i,j} = l_i$, $0 \leq l_i \leq n_i$, where, $v_{i,j} = 1$, if the jth component from the type i works and $v_{i,j} = 0$ otherwise. Let $V^i_{l_i}$ denote the set of all state vectors $v^i_{l_i}$ for components of type i. Furthermore, let $V_{l_1, l_2, ..., l_M}$ denote the set of state vectors of the form $(v_{1,1}, \ldots, v_{1,n_1}, v_{2,1}, \ldots, v_{2,n_2}, v_{M,1}, \ldots, v_{M,n_M})$ for the whole system with exactly l_1 components from the first type, l_2 components from the second type, ... and l_M components from the M-th type. In other words, $V_{l_1, l_2, ..., l_M}$ contains all vectors of the size n that fit $(\sum_{i=1}^{M} \sum_{j=1}^{n_i} v_{i,j} = l_1 + l_2 + ... + l_M)$. It is clear that $V_{l_1, l_2, ..., l_M}$ contains $\prod_{i=1}^{M} \binom{n_i}{l_i}$ number of different (vectors) as a different possible cases.

Due to the **iid** assumption for the lifetimes of the components of the same type, all these state vectors are equally likely to occur, hence

$$\phi(l_1, l_2, ..., l_M) = \frac{\sum_{\varepsilon \in V_{l_1, l_2, ..., l_M}} \Phi(\varepsilon)}{\prod_{i=1}^{M} \binom{n_i}{l_i}}. \tag{1}$$

It is clear that $\sum_{\varepsilon \in V_{l_1, l_2, ..., l_M}} \Phi(\varepsilon)$, is actually the number of permutations with exactly l_1 functioning components from the first type, l_2 functioning components from the second type, ..., and l_M functioning components from the M-th type for which the system is in the up state. In another expression, the vectors in the set $V_{l_1, l_2, ..., l_M}$ for which the system is in the up state, are somewhat similar to what so-called path vectors of the system, whereas a set of components in a system is called a path set if functioning of all components in this set implies that the system itself is functioning.

In the type i, let $C_i(t)$, denote the number of the functioning components at t. According to the **iid** assumption of the components in the same type, we have

$$P(C_i(t)=l_i)=\binom{n_i}{l_i}\overline{F}_i^{l_i}(t)F_i^{n_i-l_i}(t)$$

On the other hand, obtaining the reliability function of the system at some time instant t, requires knowing the previous probability structure for all types at the same time t. That is, obtaining the structure $P(\bigcap_{i=1,2,\dots,M} C_i(t)=l_i)$ which denotes the probability that the number of working components in the first, second, ..., and M-th type are l_1, l_2, ... and l_M respectively. From the independency between the different types of components, we have

$$\begin{aligned}P(\bigcap_{i=1,2,\dots,M}\{C_i(t)=l_i)\} &= P(C_1(t)=l_1,C_2(t)=l_2,\dots,C_M(t)=l_M)\\ &=\prod_{i=1}^{M}P(C_i(t)=l_i)\\ &=\prod_{i=1}^{M}\binom{n_i}{l_i}\overline{F}_i^{l_i}(t)F_i^{n_i-l_i}(t).\end{aligned} \quad (2)$$

Hence, one can obtain a survival-based reliability function representation of the system. Using total probability law, we have

$$P(T_s>t)=\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\dots\sum_{l_M=0}^{n_M}\phi(l_1,l_2,\dots,l_M)\times P(C_1(t)=l_1, C_2(t)=l_2,\dots,C_M(t)=l_M).$$

Using (2) we have

$$P(T_s>t)=\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\dots\sum_{l_M=0}^{n_M}\phi(l_1,l_2,\dots,l_M)\times\prod_{i=1}^{M}\binom{n_i}{l_i}\overline{F}_i^{l_i}(t)F_i^{n_i-l_i}(t). \quad (3)$$

3 Applications and illustrative examples

Wind plants are important renewable resource systems that operate to convert a portion of the kinetic energy that exists in a mass of moving air to electrical energy by way of an electric generator. A wind plant consists of a number of wind turbines, and in general, the power delivered by a wind plant P_i is the sum of the real power produced by its constituent wind turbines, Louie and Sloughter(2014)

$$P_i = \sum_{j=1}^{n_i} P_{WT,j} - P_{L,i},$$

where n_i is the number of wind turbines in the wind plant i, $P_{WT,j}$ is the real power released by the jth wind turbine and $P_{L,i}$ is the ith wind plants collector system losses at the current operating state.

As it is well known, there are several factors that can influence the progress of generating energy. Wind speed may be the most important factor in the released energy of a wind turbine. Accordingly, two turbines located in two different regions (with considerable divergence), may have different outcomes electrical energy due to the difference wind speeds between those regions. Typically, a wind power generation system consists of one or more wind plants, each plant may also contain more than one turbine. On the other hand, Louie and Sloughter deduced that the power delivered by a wind plant is stochastic and primarily depends on wind speed which treated as a random variable. They also showed that wind speed data can be modeled using the two-parameter Weibull distribution. According to this, for two wind plants located in different places far away from each other, naturally, the wind speed, and as a result, the distribution that models the wind speed, varies from one plant to the other. Whereas in the same plant the turbines are located relatively close and then the wind speed will be almost equal for all turbines, hence they all considered to have identical distribution. Here is a numerical example to illustrate the problem.

Example 3.1. Consider a wind system consists of three plants (types) located in three different regions with considerable divergence. The first plant has 5 turbines (components) with capacities (weights) in megawatts: $\{w_1^1 = 1, w_2^1 = 2, w_3^1 = 4, w_4^1 = 5, w_5^1 = 6\}$. The second plant contains three turbines with capacities in mega wats $\{w_1^2 = 3, w_2^2 = 4, w_3^2 = 8\}$, and the third plant has turbines with weights $\{w_1^3 = 9, w_2^3 = 10\}$. We consider a Weibull distribution with parameters α and β and the density function

$$f(t,\alpha,\beta) = \alpha\beta t^{\alpha-1} e^{-\beta t^{\alpha}}, \quad t > 0, \alpha > 0, \beta > 0.$$

The failure of this system occurs whenever the accumulated outcome energy from the whole system (from the three plants) becomes less than $(k = 30)$ MW, then the structure function of the system is

$$\Phi(\varepsilon(t)) = \begin{cases} 1, & \text{if } \sum_{i=1}^{3}\sum_{j=1}^{n_i} w_j^{(i)} I(\varepsilon_j^{(i)}(t) = 1) \geq 30 \\ 0, & \text{otherwise,} \end{cases}$$

Obtaining reliability function requests computing the survival signature $\phi(l_1, l_2, l_3)$ for $l_1 = 0,1,2,3,4,5$, $l_2 = 0,1,2,3$ and $l_3 = 0,1,2$. Hence, survival signature must be computed for all values of l_is i.e., for $(n_1+1)\times(n_2+1)\times(n_3+1) = 6\times 4\times 3 = 72$ different cases. The multivariate survival functioins of the sytstem is computed using a new algorithm in Table **??** where the elements

Table 1: Survival functional for the system

l_2	l_3	l_1 = 0	1	2	3	4	5
0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0
	2	0	0	1/10	3/5	1	1
1	0	0	0	0	0	0	0
	1	0	0	0	1/10	13/30	1
	2	0	1/5	2/3	29/30	1	1
2	0	0	0	0	0	0	1/3
	1	0	0	1/5	7/12	9/10	1
	2	2/3	13/15	29/30	1	1	1
3	0	0	0	0	1/10	2/5	1
	1	0	3/10	17/20	1	1	1
	2	1	1	1	1	1	1

$\phi(l_1, l_2, l_3)$, $l_1 = 0,1,2,3,4,5$, $l_2 = 0,1,2,3$ and $l_3 = 0,1,2$ refers to the probability that the system functions with exactly l_1, l_2 and l_3 components from the first, second and third type respectively. One can obtain the reliability function of the system using the elements organized in Table 1. Using (3) we have

$$P(T_s > t) = \sum_{l_1=0}^{5} \sum_{l_2=0}^{3} \sum_{l_3=0}^{2} \phi(l_1, l_2, l_3) \times \prod_{i=1}^{3} \binom{n_i}{l_i} \overline{F}_i^{l_i}(t) F_i^{n_i - l_i}(t)),$$

where $F_i(t)$ and $\overline{F}_i(t)$, $i = 1,2,3$ are the cumulative distribution function and the reliability function for the type i i.e, for Weibull distribution $W(1,2)$, $W(2,3)$ and $W(3,5)$ respectively. Reliability functions of the system in the case when $F_1(t) \sim W(2,3)$, $F_2(t) \sim W(1,2)$ and $F_3(t) \sim W(3,5)$ and in the case when $F_1(t) \sim W(2,3)$, $F_2(t) \sim W(3,5)$ and $F_3(t) \sim W(1,2)$ are plotted in Figure 1.

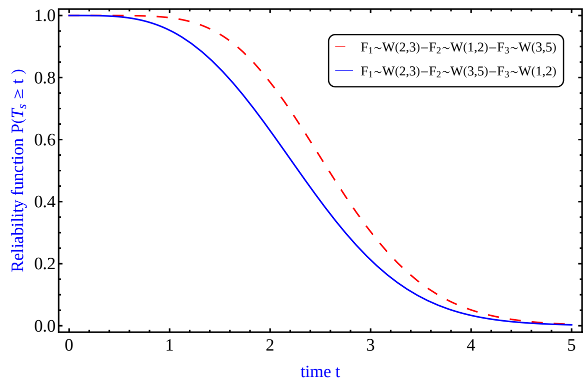

Figure 1: Reliability functions of the system for the two different cases.

Figure 1 shows that in the case of $F_1(t) \sim W(2,3)$, $F_2(t) \sim W(1,2)$ and $F_3(t) \sim W(3,5)$, we have a more reliable system than the second case.

Furthermore, reliability functions of the system with all possible cases of the distributions between the three types with a system threshold $k = 30$ are plotted in Figure 2 where all these possible cases are organized in Table 2

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Type 1	W(1,2)	W(1,2)	W(2,3)	W(2,3)	W(3,5)	W(3,5)
Type 2	W(2,3)	W(3,5)	W(1,2)	W(3,5)	W(1,2)	W(2,3)
Type 3	W(3,5)	W(2,3)	W(3,5)	W(1,2)	W(2,3)	W(1,2)

Table 2: All possible cases for the three types distributions

It is clear from Figure 2 that the most reliable system we will have is in

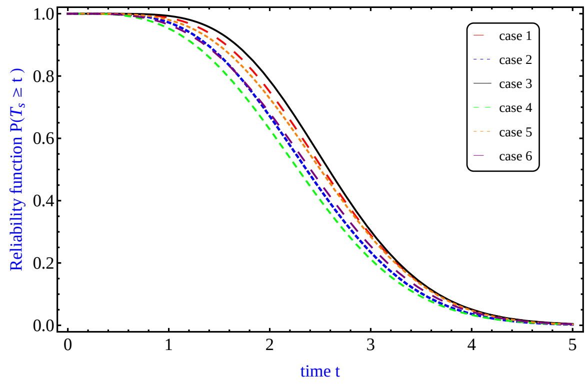

Figure 2: Reliability functions for all cases.

the case of $F_1(t) \sim W(2,3)$, $F_2(t) \sim W(1,2)$ and $F_3(t) \sim W(3,5)$ whereas the worse case is when $F_1(t) \sim W(2,3)$, $F_2(t) \sim W(3,5)$ and $F_3(t) \sim W(1,2)$. All the other cases are between these two cases as the above figure illustrates.

4 Conclusions

In the present paper, we introduced the concept of extended survival signature for the case of weighted k-out-of-n systems. Assuming that each component in the system has its own capacity (weight) and the system starts operating at time $t = 0$. It was assumed that the failure of the system occurs when the total weight of the functioning components becomes less than a predetermined threshold k. The reliability function of the system lifetime is represented in terms of extended survival signature. Some illustrated examples were also provided.

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Optimal Maintenance Strategies for a Warranted Coherent System

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Abstract: In this talk, we propose a maintenance model for a warranted coherent system. The warranty period has two phases under which the manufacturers commitment comes in two different forms. In Phase I, upon the system failure, the failed components are replaced and a corrective maintenance is conducted on the system. If the system failure occurs during Phase II, only a minimal repair is performed on the whole system. Following the expiration of warranty, the customer replaces the failed components and preventively maintained the system at the end of such a maintenance period. During the maintenance period, a generalization of age-based maintenance model is conducted on the system and components. The main goal is to determine, from the customers perspective, the optimal planned time of preventive maintenance in the maintenance period. A numerical example is provided to illustrate the proposed optimal maintenance model.

Keywords: Minimal repair, Preventive maintenance, Renewing warranty, Signature, Twophase warranty.

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1 Introduction

Nowadays, due to advancement of technology and very competitive markets, the manufacturers are under pressure to provide better services for their products during a pre-specified time period, referred to as the warranty period. Thus, almost all products are sold by offering a long-term warranty. The warranty is a contract between the product manufacturer (vendor or seller) and the customer (buyer). The manufacturer assures that, in the case of product failure under normal usage, it will repair, replace or provide pre-defined compensation to the buyers within the warranty duration ([17] and [19]). Considering the characteristics of products such as the products complexity, reliability and repairability, the manufacturer can offer different types of warranties ([11]).

Among various types of warranty, there exist two general types of warranty policies: renewing warranty and non-renewing warranty. Under a renewing warranty, a failed product within the warranty duration is replaced by a new one, and the warranty is renewed at no charge to the customer or at a partial cost to the customer. Under a non-renewing warranty, the failed product is satisfactorily serviced only within the initial warranty period. That is, when the product has failed, it is replaced or repaired at no cost to the customer or at a certain cost to the customer and the warranty is not renewed.

Usually, the customer is mainly interested in developing the optimal maintenance strategy after the warranty expires. Maintenance strategies play an essential role in retaining product reliability, availability and quality at a suitable level. In the literature, maintenance is generally grouped into two main types: corrective maintenance (CM) and preventive maintenance (PM). The CM is an unplanned action that is performed after products failure and restores it to an operational state, whereas the PM is an action which is performed at a planned time before products failure. According to degree of repair of the product, the maintenance actions are also classified into two main categories. The first type of maintenance action is known as minimal which is a repair action that restores the system to the working state identical to that of before failure. The second one is the perfect repair under which the system returns to as-good-as-new state. A large number of research works are devoted to various maintenance schedules. Among many references, we refer the reader to [4], [18], [10], [12] and [5].

At the beginning of the 21th century, both warranty and maintenance have received increasing attention. In the literature, some remarkable research works have also been appeared that links warranty and maintenance. [3] considered an age-replacement model in which incorporates minimal repair for products under a renewing free-replacement warranty policy. The impacts of a product warranty on the optimal replacement model are also investigated. [15] presented a literature review for the articles that link warranty and maintenance. [16] proposes, from the consumers perspective, a replacement policy after the expiry of the warranty, under the renewable free replacement warranty policy in which the replacement is dependent on the repair-cost threshold. For more references on warranty and maintenance, we refer, among others, to [13], [8], [7], [9].

In the present paper, we are mainly interested in developing the optimal maintenance models using the signature-based reliability representation of the system lifetime. This representation enables us to propose cost-based optimal models such that at each time in which the maintenance action should be performed, the repairs/replacements are carried out not only on the entire system but also on each component. Also, for a coherent system sold with warranty, we consider the optimal maintenance strategies from the customers perspective after the warranty expires.

The arrangement of the paper is as follows: In Section 2, we first give the mixture representation of the system reliability function based on the concept of signature. We then propose a maintenance model for a warranted coherent system. To illustrate the results of the paper, a numerical example is provided in Section 3.

2 General maintenance model following warranty period

In assessing the reliability and stochastic properties of coherent systems, an approach, which has received great attention, is to use the notion of signature. For a coherent system (see, [2]) with n independent and identically distributed components, let $X_1, X_2, ..., X_n$ denote the lifetimes of components with cumulative distribution function F and probability density function f. The reliability function $\bar{H}(t) = P(T > t)$ of the system's lifetime $T = \tau(X_1, , X_n)$ is given as ([14])

$$\bar{H}(t) = \sum_{i=1}^{n} s_i(1 - F_{i:n}(t))$$
$$= \sum_{i=0}^{n-1} \bar{S}_i \binom{n}{i} F^i(t)(1 - F(t))^{n-i},$$

where $X_{i:n}$ is the ith smallest among $X_1, X_2, , X_n$ with cumulative distribution function $F_{i:n}(t)$, $s_i = P(T = X_{i:n})$, $i = 1, 2, ..., n$ and $\bar{S}_i = \sum_{j=i+1}^{n} s_i$. The probability vector $s = (s_1, s_2, , s_n)$ is known as the signature vector of the system. The ith element of s is, in fact, the probability that the ith component failure causes the system failure. It is calculated as $s_i = n_i/n!$, where n_i denotes the number of permutations of components under which the ith component failure causes the system failure.

Now, assume that a new coherent system consisting of n components begins to operate at time $t = 0$. The system manufacturer offers a warranty duration of length w at the beginning of system operation. The warranty period $(0, w)$ is separated into two non-overlapping subintervals $(0, \alpha w)$ and $(\alpha w, w)$ with $0 \leq \alpha \leq 1$, referred respectively to as Phase I and Phase II. The two-phase warranty considered in this section works as follows. When the system failure occurs during Phase I, the failed components are replaced and a CM are performed on the whole system, and the warranty is renewed with the original warranty terms. If the failure occurs during the Phase II, only a minimal repair is conducted on the whole system by the manufacturer.

If there is no Phase I, that is, $\alpha = 0$, then the manufacturer only performs minimal repair on the system during the whole warranty period $(0, w)$. On the other hand, if $\alpha = 1$, Phase II is omitted and the failed system is rectified by replacing the failed components and CM on the system during the entire warranty period $(0, w)$.

When the two-phase warranty expires, the customer takes over the full responsibility for maintaining the system. As a maintenance model following the warranty period, we consider an age-based model. If the system fails during the interval $(w, w + T_{PM})$, the failed components are replaced and a CM are performed on the whole system; but if the system is still alive until $w + T_{PM}$, the failed components are again replaced and the operator performs a PM on the entire system. Here, T_{PM} is a planned fixed time. It is assumed that both CM and PM are perfect so that the system will be as-good-as-new.

After the warranty expires, the customer is fully responsible for the cost incurred during the life cycle of the system. In order to improve the system operation, the customer may conduct a preventive maintenance strategy on the system/components. So far, a huge number of maintenance models for systems have been appeared in the literature. In this section, an age-based maintenance policy among various existing maintenance models is applied on the system beyond the expiration of the warranty and the expected cost rate is evaluated from the customers perspective. In an age-based maintenance strategy, each maintenance period ends at age $T_{PM} + w$ or whenever the system fails, whichever occurs first.

Let $T_w = (T - w \mid T > w)$ be the residual lifetime of the system when the warranty expires. The reliability function of T_w is given by

$$\bar{H}(x \mid w) = \frac{\bar{H}(w+x)}{\bar{H}(w)}, \quad x \geq 0.$$

Suppose that M denotes the number of system failures during the Phase I warranty of length αw. Therefore, the random variable M has the following geometric distribution,

$$P(M = m) = \bar{H}(\alpha w) H^m(\alpha w), \qquad m = 0, 1, 2, \ldots$$

with the expected value $H(\alpha w)/\bar{H}(\alpha w)$. Let T_i, $i = 12,...$, where $T_i < \alpha w$, denote the lifetimes of the systems consecutively installed during the time interval $(0, \alpha w)$. Assume that $\mathbb{S}(T_{PM})$ is the duration of each life cycle under our proposed maintenance model including the warranty term. The expected length of life cycle can then be obtained as

$$\begin{aligned} E[\mathbb{S}(T_{PM})] &= E\left(\sum_{j=1}^{M} T_j\right) + w + \min(T_{PM}, T_w) \\ &= E(M)E(T_1 \mid T_1 < \alpha w) + w + \int_0^{T_{PM}} \frac{\bar{H}(w+x)}{\bar{H}(w)} dx \\ &= \int_0^{\alpha w} \frac{xh(x)}{\bar{H}(\alpha w)} dx + w + \int_0^{T_{PM}} \frac{\bar{H}(w+x)}{\bar{H}(w)} dx. \end{aligned}$$

Under such a maintenance model, whenever the age of the system reaches planned time $T_{PM} + w$, the failed components are replaced by new ones at a cost c_R and a PM action is performed on the system at a cost c_{PM}. On the other hand, if the system fails in the interval $(w, w + T_{PM})$, in a similar manner, the failed components are replaced by new ones at a cost c_R and a CM action is performed on the system at a cost c_{CM}. In addition, during the warranty period, we consider a system failure cost at a cost c_F which is the responsibility of the customer.

The costs of components replacement, CM and minimal repairs are free of charge for the customer on the warranty period $(0, w)$. Suppose that $\mathbb{C}(T_{PM})$ denotes the total cost incurred by the customer. Then, the expected total cost, from the viewpoint of the customer, in the warranty period and the maintenance period is

$$\begin{aligned} E[\mathbb{C}(T_{PM})] &= (c_R E[N_{T_{PM}+w} \mid T > T_{PM} + w] + c_{PM}) P(T_w > T_{PM}) \\ &\quad + (c_R E[N_T \mid w < T < T_{PM} + w] + c_{CM}) P(T_w < T_{PM}) \\ &\quad + c_F E(M) + c_F \int_{\alpha w}^{w} \lambda(t) dt, \end{aligned}$$

where N_t is the number of failed component of the system at the time t. Using a result in [1], we have

$$E[N_t \mid T > t] = \frac{\sum_{i=1}^{n-1} i\bar{S}_i\binom{n}{i}F^i(t)\bar{F}^{n-i}(t)}{\bar{H}(t)}.$$

Also, it can be shown that

$$E[N_T \mid w < T < w+t] = \frac{\sum_{i=1}^{n} is_i[F_{i:n}(w+t) - F_{i:n}(w)]}{H(w+t) - H(w)}.$$

Thus

$$\begin{aligned} E[\mathbb{C}(T_{PM})] &= c_F\left(\frac{H(\alpha w)}{\bar{H}(\alpha w)} + \int_{\alpha w}^{w} \lambda(t)dt\right) \\ &+ \left(c_R\frac{\sum_{i=1}^{n-1} i\bar{S}_i\binom{n}{i}F^i(T_{PM}+w)\bar{F}^{n-i}(T_{PM}+w)}{\bar{H}(T_{PM}+w)} + c_{PM}\right)\frac{\bar{H}(T_{PM}+w)}{\bar{H}(w)} \\ &+ \left(c_R\frac{\sum_{i=1}^{n} is_i[F_{i:n}(T_{PM}+w) - F_{i:n}(w)]}{H(T_{PM}+w) - H(w)} + c_{CM}\right)\left(1 - \frac{\bar{H}(T_{PM}+w)}{\bar{H}(w)}\right). \end{aligned}$$

Therefore, the long-run expected cost per unit of time for the coherent system under the proposed maintenance model is (see, e.g. Ross, 1996)

$$\eta(T_{PM}) = \frac{E[\mathbb{C}(T_{PM})]}{E[\mathbb{S}(T_{PM})]}. \tag{1}$$

In the following, we specify the optimal value T^*_{PM} minimizing the expected cost rate from the customer's perspective. It may be interested to determine the optimal value T^*_{PM} which results in the minimum value for the expected cost rate. In other words, the optimization problem can be formulated as minimizing the expected cost rate $\eta(T_{PM})$, given in Equation (1), with respect to the decision variable T_{PM}.

In the next proposition, sufficient conditions for the existence of the optimal value T^*_{PM} are provided. The proof can be found in [6].

Proposition 2.1. *Let the signature vector s be such that $(n-i)s_i/\bar{S}_i$ is increasing in i, and $\eta(T_{PM})$ be as given in Equation (1). Assume that $n^* = \max\{i : s_i > 0\}$. Let also $m(w)$ be the mean residual life function of the system at w. If*

$$\left[(n^*-1)^2 c_R\tilde{r}(w) + (n-n^*+1)(c_{CM} - c_{PM} + n^* c_R)r(w)\right]\left(w + \frac{\int_0^{\alpha w} xh(x)dx}{\bar{H}(\alpha w)}\right) < c_{PM},$$

and

$$\lambda(\infty) > \frac{(n^*-1)c_R + c_{CM} + (H(\alpha w)/\bar{H}(\alpha w) + \int_{\alpha w}^{w} \lambda(x)dx)c_F}{(c_{CM} - c_{PM} + (n-n^*+1)c_R)(w + m(w) + \int_0^{\alpha w} xh(x)dx/\bar{H}(\alpha w))},$$

then there exists a finite optimum value T_{PM}^* *that minimizes* $\eta(T_{PM})$.

3 Numerical example

This section presents a numerical example to illustrate the proposed optimal maintenance model. Let us consider the coherent system depicted in Figure 1. The system's signature is computed, using a Mathematica program, as

$$s = \left(0, \frac{1}{45}, \frac{37}{360}, \frac{57}{280}, \frac{163}{630}, \frac{143}{630}, \frac{19}{140}, \frac{1}{20}, 0, 0\right).$$

The lifetimes of components are assumed to be independent and have a common Weibull distribution with failure rate $\lambda(t) = 0.1163\sqrt{t}$. Suppose that the parameter values we set for this particular numerical example are as follows: The replacement cost for each failed component is $c_R = 30$ and the length of two-phase warranty period is $w = 1.5$.

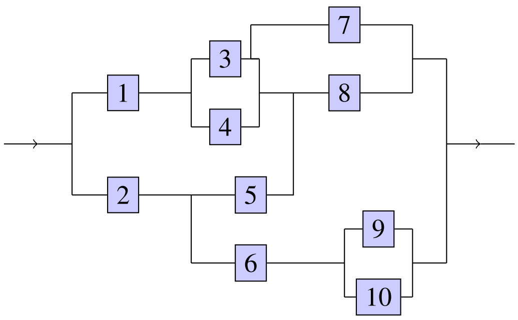

Figure 1: A system with 10 components.

Tables 1 contains the optimum values T_{PM}^* and the optimum cost $\eta(T_{PM}^*)$ from customer's perspective. An analysis of the results of this table indicates that when c_{CM} gets larger, then, as expected, the optimal time T_{PM}^* of PM decreases. In other words, the PM time should be earlier to prevent system failure and the customer incur less maintenance costs. Also, it is observed that an increase in c_{CM} results in an increase in $\eta(T_{PM}^*)$. It is seen that when

c_{PM} increases, then the optimal time of PM gets larger, too. This means that an increase in the cost of preventive maintenance of the system makes the customer to postpone the time of PM action.

Table 1: The optimal T^*_{PM} and $\eta(T^*_{PM})$ for different values of c_{PM} and c_{CM} with $c_R = 30$, $c_F = 10$, $p = 0.8$, $\alpha = 0.4$ and $w = 1.5$.

	c_{PM}					
	200		210		220	
c_{CM}	T^*_{PM}	$\eta(T^*_{PM})$	T^*_{PM}	$\eta(T^*_{PM})$	T^*_{PM}	$\eta(T^*_{PM})$
330	2.752	109.274	3.095	110.431	3.519	111.332
340	2.588	110.702	2.889	112.002	3.248	113.056
350	2.448	112.058	2.214	113.491	3.027	114.687

The plot of maintenance cost per unit time from customer's perspective is depicted in Figure 1 for $c_F = 10$, $c_R = 30$, $c_{CM} = 340$, $c_{PM} = 220$, $w = 1.5$ and $\alpha = 0.4$.

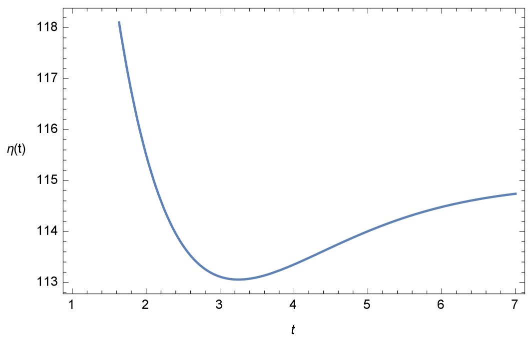

Figure 2: The maintenance cost function with $p = 0.8$.

4 Conclusions

For a warranted coherent system, it is of essential importance to establish optimal maintenance strategies after the warranty expires. The purpose of this paper was to propose an optimal preventive maintenance model for warranted coherent systems. The warranty period was separated into two non-overlapping periods, named Phase I and Phase II. We utilized the expected cost rate per unit time as a criterion of the optimality. We also considered sufficient conditions for the existence and uniqueness of the optimal decision variable. The results were developed by using the signature-based reliability representation of the system lifetime. In the proposed models, we concentrated on the case where the lifetimes of components are independent and identically distributed. It is interesting to consider, as a future study, the case of dependent and/or non-identical components which is a more realistic assumption in some practical situations.

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Optimal Warranty Plan with Limited Number of Repairs Based on a Rate Cost Function

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Abstract: One of the important issues in a warranty plan issued by sellers or in an insurance plan undertaken by insurance companies is determining the plan duration and the frequency of service provision in an attempt to minimize the costs or maximize the profits. In this paper, we determined the optimal extended warranty length with limited number of repairs in the warranty duration. For this purpose, a rate cost function is introduced and the optimal values of the warranty plan are obtained based on it.

Keywords: Minimal repair, Optimization, Repairable system, Rate cost function, Warranty period.

1 Introduction

Due to advances in technology and the need to produce and use complex industrial systems, consumers of these products are always concerned about the rising purchasing costs or reliability of the purchased product. Manufacturers also want to increase the number of customers of their products and the profit from their sales. Therefore, a warranty is considered to be a contract or agreement between the seller and the buyer that starts at the moment of purchase of the product and continues until the end of the warranty period. So one of the

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important issues in a warranty plan issued by sellers or in an insurance plan undertaken by insurance companies is determining the plan duration and the frequency of service provision in an attempt to minimize the costs or maximize the profits. For this purpose, various types of warranties have been introduced and discussed so far. For instance, free repair warranty (FRW) and pro-rata warranty (PRW) are the most popular classes of warranty policies which have received much attention.

In general, two types of repair action can be taken to repair a repairable system within the warranty period: perfect repair and minimal repair, and the system state depends on the type of repair implemented. [4] introduced an extended warranty policy that includes a FRW period and an extended warranty period. Then, they obtained the optimal policies for the consumers and the manufacture. [2] studied a repair cost limitation during a fixed warranty period and assumed that if the cost of repairs during the warranty period is more than a fixed amount, minimal repairs will no longer be performed. [1] developed lifetime warranty policies and models for predicting failures and estimating costs for lifetime warranty policies. [6] introduced a new two-dimensional (2-D) warranty policy with respect to the failure time and warranty servicing time. [7] studied the consumer's and the manufacturer's optimal strategies for items sold with periodic PM under a 2-D warranty policy. [5] considered an optimal periodic PM policy after the expiration of 2-D warranty. [8] investigated a multi-phase reliability growth test program for repairable products that sold with a 2-D warranty. In this paper, we proposed an extended two-dimensional warranty policy which includes limitation on time and the number of repairs. Then, using an appropriate cost function, we will find the optimal values of the extended warranty period and the number of minimal repairs.

In Section 2, model description and the cost function are explained. The functions required to define the cost function are presented in Section 3. The numerical optimization results are studied in Section 4.

2 Model assumptions and cost function

Let us consider a repairable system that can be minimally repaired after each failure and sold with an extended warranty model. The product is sold with an initial warranty period and an extended warranty period with the condition of limiting the number of minimal repairs. The proposed extended warranty period is such that the manufacturer agrees to repair at most n minimal repairs in a limited warranty period $[0,w]$. If the number of repairs in the warranty period $[0,w]$ is less than n failures, the warranty period will extend up to the nth failure occurs or up to a prefixed time τ is passed. It means that the duration of the company's services or the warranty length is stopped at $T_w = \min\{T_n, \tau\}$, where T_n is the time at which the nth minimal repair is done. We carry out our study under the following assumptions and notation.

1) A new system is put into the operation at time $t = 0$.
2) All failures are detected immediately, and the repair times are negligible.
3) A minimal repair does not change the failure rate.
4) All repair costs are charged by the manufacturer in the period $[0,T_w]$.
5) The consumer will have to pay a fee proportional to the ratio of the remaining repairs to the duration of the extended warranty length.

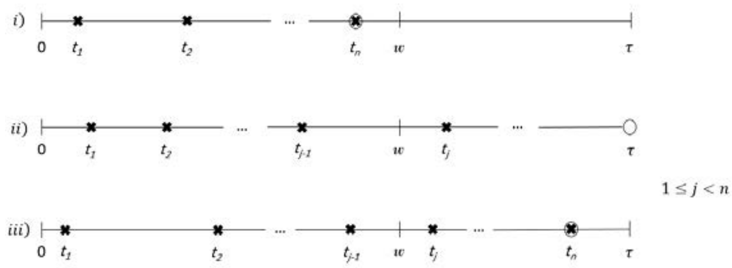

Figure 1: The graph of the proposed plan, where $\times$ and $\bigcirc$ denote the failure times and the end of the warranty period, respectively.

The purpose of this study is to find the optimal values of n and τ by minimizing the costs of the manufacturer. Note that this warranty plan is introduced by [3]. They use two cost functions to find the optimal values of the plan. But, in this paper, we use the rate of cost associated with the duration of system operation. With this in mind, suppose that $N(t)$, $E(N(t))$ and $E(T_w)$ are the number of minimal repairs in $(0,t)$, expected number of minimal repairs and expected duration of the warranty, respectively. Therefore, the rate of costs is considered as follows

$$RC_w(n,\tau) = -C_0 + C_1 E(N(T_w)) - C_2 \frac{E\left(N(T_w - w)|T_w > w\right)}{E\left(T_w|T_w > w\right)}, \quad (1)$$

where $C_0 = C_0'' - C_0'$, C_0' is cost of the system production for the manufacturer and C_0'' is the initial purchase price for the consumer. C_1 is the cost of each minimal repair paid by the manufacturer. It includes labor, administrative costs, etc. C_2 is the cost paid by the consumer to extend the warranty period.

Let $f(.)$ and $F(.)$ be the probability density function (pdf) and cumulative distribution function (cdf) of the lifetime of the original system, respectively. By the assumptions, $\{N(t); \quad t \geq 0\}$ can be verified as an nonhomogeneous Poisson process (NHPP). Then, the probability mass function (pmf) of $N(t)$ is given by

$$P\left(N(t) = n\right) = \frac{\left(\Lambda(t)\right)^{n-1}}{(n-1)!} \exp\left(-\Lambda(t)\right), \qquad n = 1, 2, \ldots, \quad (2)$$

where $\Lambda(t) = -\log\left(1 - F(t)\right)$ is the cumulative failure rate of the original system or the cumulative ROCOF of the repairable system. Also, the cdf of T_n is given by

$$F_{T_n}(t) = 1 - \sum_{i=0}^{n-1} \frac{\left(\Lambda(t)\right)^i}{i!} \exp\left(-\Lambda(t)\right). \quad (3)$$

By using (3), the survival function of $T = \min\{T_n, \tau\}$ (i.e., $\bar{F}_{T_w}(t) = 1 - F_{T_w}(t)$) is given by

$$\bar{F}_{T_w}(t) = \begin{cases} \bar{F}_{T_n}(t), & t \leq \tau, \\ 0, & t > \tau. \end{cases} \quad (4)$$

From (4) by positivity of T, the conditional expected value of T is

$$E\left(T_w|T_w > w\right) = E\left(\min\{T_n, \tau\}|T_n > w\right) = \frac{1}{\bar{F}_{T_n}(w)} \int_w^\tau \bar{F}_{T_n}(x)dx + w. \quad (5)$$

On the other hand, using (3) and (3), the expected number of minimal repairs in $[0, T_w]$ can be written as follows

$$E\left(N(T_w)\right) = nF_{T_n}(\tau) + \Lambda(\tau)\left(1 - F_{T_{n-1}}(\tau)\right). \tag{6}$$

The expected number of minimal repairs in the extended warranty period is given by

$$E(N(T_w - w)|T_w > w) = (n - \Lambda(w))\left(F_{T_n}(\tau) - F_{T_n}(w)\right) + \left(\Lambda(\tau) - \Lambda(w)\right)\bar{F}_{T_n}(\tau). \tag{7}$$

Therefore, numerical methods need to be used to obtain the optimal values of the warranty plan. So, we use the algorithm that proposed by [3].

3 Numerical results

To illustrate the results in the previous sections, we present some graphical and numerical computations. Throughout this section, assume that the lifetime of the original system follows the Weibull distribution with $\lambda(t) = bt^{b-1}$, $b > 0, t > 0$. It is obvious that in this case, the ROCOF is increasing in t for $b > 1$ and decreasing in t for $b < 1$. Also, we assume that $C_0 = 30$, $C_1 = 50$, $C_2 = 150$ and $w = 1$. Figure 2 shows that the rate cost function (2) has an optimal solution in n and τ when $b = 2$. From Figure 3, it is obvious that the minimum of this cost function is unique in τ for some selected values of n. In Figure 4, it is shown that the rate cost function (5) has an optimal solution in n for some selected values of τ.

We have determined the optimal values of (n^*, τ^*) and $RC_w(n^*, \tau^*)$ for different values of b and w. The results are presented in Tables 1 and 2, respectively.

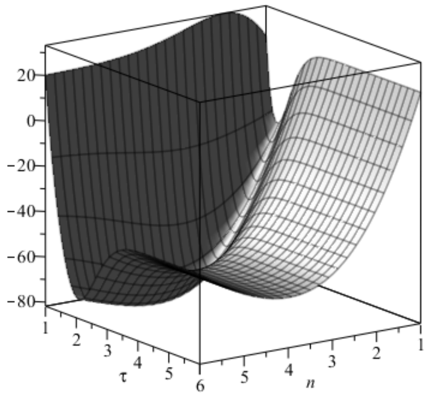

Figure 2: Plot of $RC_w(n,\tau)$.

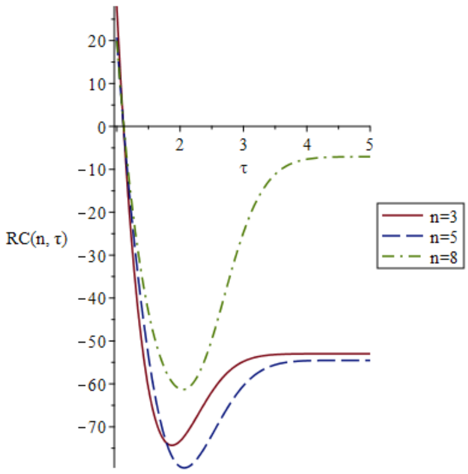

Figure 3: Plot of $RC_w(n,\tau)$ for selected value of n.

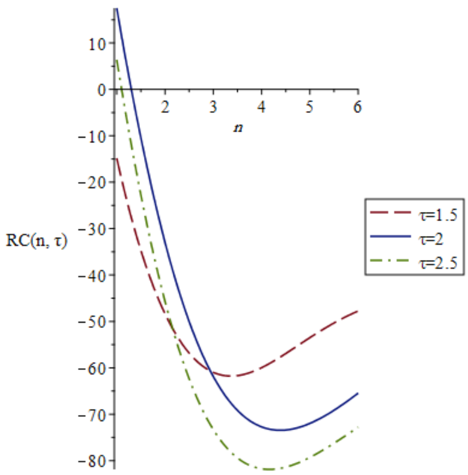

Figure 4: Plot of $RC_w(n,\tau)$ for selected value of τ.

Table 1: Optimum values of (n^*,τ^*) and $RC_w(n,\tau)$

b	(n^*,τ^*)	$RC_w(n^*,\tau^*)$
2	(5, 2.1726)	-122.3465
2.5	(7, 2.1611)	-176.1641
3	(11, 2.2173)	-252.223
3.5	(19, 2.3138)	-365.4935

$(C_0,C_1,C_2) = (30,50,150)$ and $w = 1$.

Table 2: Optimum values of (n^*,τ^*) and $RC_w(n,\tau)$

w	(n^*,τ^*)	$RC_w(n^*,\tau^*)$
1	(5, 2.1726)	-122.3465
1.5	(6, 2.4119)	-28.4276
2	(8, 2.7712)	87.2371

$(C_0,C_1,C_2) = (30,50,150)$ and $b = 2$.

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Systems with Two-dependent Components under Spare Switching Policy

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Abstract: Spares are commonly used to improve system performances. They are allocated to original components in parallel during system missions. Various methods for finding optimal allocations have been proposed in the literature. The optimal configurations depend on system structures and component lifetimes. For sake of brevity, lifetimes of components are commonly assumed to be independent. This paper deals with systems when component lifetimes are dependent and heterogeneous. Moreover, the spare is also allowed to switch among original components in order to impose more flexibility for spare management. Explicit expressions for system reliability functions are derived in details. Since system lifetimes are random phenomena, stochastic orders are utilized for comparison purposes. various illustrative examples are also given.

Keywords: Reliability, Redundancy, Stochastic orders, Switching.

1 Introduction

Additional components (spares) are used to improve engineering system performances. For more details, see Barlow and Proschan [2], Nakagawa [15].

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There are many papers deal with redundancy allocation problem in reliability systems. Boland et al. [5] applied stochastic orders to consider this problem for series and parallel systems. Zhao et al. [22, 21] studied optimal allocation of redundancies with exponential components in the sense of various stochastic orders. Xie et al. [20] investigated the redundancy allocation problem in k-out-of-n hot standby systems to maximize the operational availability. But in the case of dependent components, there are not many works. Among a few works, Navarro et al. [17] studied the performance of a system composed by different kinds of units maybe having dependent lifetimes to evaluate reliability. Navarro and Durante [16] studied the behaviour of the residual lifetimes of coherent systems with possibly dependent components. Belzunce et al. [3, 4] used the concept of joint stochastic orders and Jeddi and Doostparast [8, 9] studied this problems for series and parallel systems. Redundants are allocated to original components during system missions. Commonly, spares do not switch among the original components. This restriction usually exist in industrial systems. Meanwhile in some applications such as networks, system managers may be able to control and switch spares among original components to achieve more reliable systems. In other words, redundants can change dynamically their respective original components; For more examples and recent developments, see Kim et al. [12], Li et al. [14], Jia et al. [11] and references therein. Notice that, the spare can not switch if the corresponding original component fails. In the sequel, let $(\Omega, \mathbb{F}, p)$ be a probability space and $\mathbf{X} = (X_1, \cdots, X_k) : \Omega \to \mathbb{R}^{+k}$, for $k \geq 1$, be an absolutely continuous random vector with the joint distribution (survival) function $F_{X_1,\cdots,X_k}(a_1, \cdots, a_k) = P(X_1 \leq a_1, \cdots, X_k \leq a_k)(\overline{F}_{X_1,\cdots,X_k}(a_1, \cdots, a_k) = P(X_1 > a_1, \cdots, X_k > a_k))$ for all $(a_1, \cdots, a_k) \in \mathbb{R}^{+k}$. Then, the density function of $\mathbf{X}$ is given by $f_{X_1,\cdots,X_k}(a_1, \cdots, a_k) = \partial F_{X_1,\cdots,X_k}(a_1, \cdots, a_k)/(\partial a_1 \cdots \partial a_k)$. The marginal distribution of $X_i (1 \leq i \leq k)$ is denoted by $F_{X_i}(x) = P(X_i \leq x), \forall x \in \mathbb{R}^{+k}$. The random variabels X_i is said to be smaller than $X_j (j \neq i)$ in usual stochastic order denoted by $X_i \leq_{st} X_j$ if $F_{X_i}(x) \geq F_{X_j}(x), \forall x \in \mathbb{R}^{+k}$. Equivalantly, $\overline{F}_{X_i}(x) \leq \overline{F}_{X_j}(x)$ where $\overline{F}_{X_i}(x) = 1 - F_{X_i}(x)$ for $1 \leq i \leq k$; See Shaked and Shanthikumar [19].

This paper is organized as follows. In Section 2, a general form for the system reliability function is presented under two above-mentioned schemes. The main result for systems with heterogeneous and dependent component lifetimes are given and proved in Section 3. Section 4 deals with some special cases which simplify the obtained general findings. Section 5 concludes the paper.

2 Building and deriving reliability function

Consider a 2-component series system consisting of a spare which can be added to the system configuration. The spare can switch only one-time. Let $\tau > 0$ be a preassigned deterministic constant and $T_i^{[0,\tau]} (i = 1, 2)$ denote the system lifetime when the spare is allocated (in parallel) to Component i during interval $[0, \tau]$ and then to Component $j(\neq i)$ beyond $\tau(> 0)$. In sequel, the reliability function of the system is derived when the spare allows to switch among the system components. To do this, let X_1 and X_2 denote the component lifetimes, and S stands for the spare lifetime. The system lifetime without the spare is $\wedge(X_1, X_2)$. As above mentioned, the system can be improved by utilizing the spare. Then the lifetime of the improved system, that is $T_1^{[0,\tau]}$ is given by

$$
\begin{aligned}
T_1^{[0,\tau]} &= \begin{cases} \wedge(\vee(X_1,S),X_2), & \text{if } X_1 \leq \tau, \\ X_2, & \text{if } X_1 > \tau, X_2 \leq \tau, \\ \tau + \wedge(X_1 - \tau, X_2 - \tau), & \text{if } X_1 > \tau, X_2 > \tau, S \leq \tau, \\ \tau + \wedge(X_1 - \tau, \vee(X_2 - \tau, S - \tau)), & \text{if } X_1 > \tau, X_2 > \tau, S > \tau, \end{cases} \\
&= \begin{cases} \wedge(\vee(X_1,S),X_2), & \text{if } X_1 \leq \tau, \\ X_2, & \text{if } X_1 > \tau, X_2 \leq \tau, \\ \tau + \wedge(X_1 - \tau, \vee(X_2 - \tau, S - \tau)), & \text{if } X_1 > \tau, X_2 > \tau, \end{cases}
\end{aligned}
$$

$$= \begin{cases} \wedge(\vee(X_1,S),X_2), & \text{if } X_1 \le \tau, \\ X_2, & \text{if } X_1 > \tau, X_2 \le \tau, \\ \wedge(X_1,\vee(X_2,S)), & \text{if } X_1 > \tau, X_2 > \tau, \end{cases} \tag{1}$$

where $\vee(a_1,a_2) = \max\{a_1,a_2\}$ and $\wedge(a_1,a_2) = \min\{a_1,a_2\}$. Similarly

$$T_2^{[0,\tau]} = \begin{cases} \wedge(X_1,\vee(X_2,S)), & \text{if } X_2 \le \tau, \\ X_1, & \text{if } X_2 > \tau, X_1 \le \tau, \\ \wedge(\vee(X_1,S),X_2)), & \text{if } X_2 > \tau, X_1 > \tau. \end{cases} \tag{2}$$

Equations (1) and (2) can be simplified after some algebraic manipulations, as

$$\begin{aligned} T_1^{[0,\tau]} &= \wedge(\vee(X_1,S),X_2)I(X_1 \le \tau) + X_2 I(X_1 > \tau, X_2 \le \tau) \\ &+ \wedge(X_1,\vee(X_2,S))I(X_1 > \tau, X_2 > \tau), \end{aligned} \tag{3}$$

and

$$\begin{aligned} T_2^{[0,\tau]} &= \wedge(X_1,\vee(X_2,S))I(X_2 \le \tau) + X_1 I(X_2 > \tau, X_1 \le \tau) \\ &+ \wedge(\vee(X_1,S),X_2))I(X_2 > \tau, X_1 > \tau), \end{aligned} \tag{4}$$

where $I_A(t)$ denotes the indicator function on the set A, i.e., $I_A(t) = 1$ for $t \in A$, and $I_A(t) = 0$ otherwise. From Equation (3), we have for $0 < t < \tau$,

$$\begin{aligned} P(T_1^{[0,\tau]} > t) &= P(T_1^{[0,\tau]} > t, X_1 \le \tau) + P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 \le \tau) \\ &+ P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 > \tau) \\ &= P(\wedge(\vee(X_1,S),X_2) > t, X_1 \le \tau) + P(X_1 > \tau, X_2 \le \tau, X_2 > t) \\ &+ P(\wedge(X_1,\vee(X_2,S)) > t, X_1 > \tau, X_2 > \tau) \\ &= P(\vee(X_1,S) > t, X_1 \le \tau, X_2 > t) + P(X_1 > \tau, t < X_2 \le \tau) \\ &+ P(X_1 > t, \vee(X_2,S) > t, X_1 > \tau, X_2 > \tau) \\ &= P(X_1 \le \tau, X_2 > t) - P(X_1 \le \tau, X_2 > t, X_1 \le t, S \le t) \\ &+ P(X_1 > \tau, X_2 > t) - P(X_1 > \tau, X_2 > \tau) \\ &+ P(X_1 > \tau, X_2 > \tau) - P(X_1 > \tau, X_2 > \tau, X_2 \le t, S \le t) \\ &= P(X_1 \le \tau, X_2 > t) - P(X_1 \le t, X_2 > t, S \le t) \\ &+ P(X_1 > \tau, X_2 > t) \end{aligned}$$

$$
\begin{aligned}
&= P(X_2 > t) - P(X_1 \le t, S \le t) + P(X_1 \le t, X_2 \le t, S \le t) \\
&= 1 - F_{X_2}(t) - F_{X_1,S}(t,t) + F_{X_1,X_2,S}(t,t,t). \qquad (5)
\end{aligned}
$$

Similarly for $t \ge \tau$,

$$
\begin{aligned}
P(T_1^{[0,\tau]} > t) &= P(T_1^{[0,\tau]} > t, X_1 \le \tau) + P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 \le \tau) \\
&+ P(T_1^{[0,\tau]} > t, X_1 > \tau, X_2 > \tau) \\
&= P(\wedge(\vee(X_1,S),X_2) > t, X_1 \le \tau) + P(X_1 > \tau, X_2 \le \tau, X_2 > t) \\
&+ P(\wedge(X_1, \vee(X_2,S)) > t, X_1 > \tau, X_2 > \tau) \\
&= P(\vee(X_1,S) > t, X_1 \le \tau, X_2 > t) \\
&+ P(X_1 > t, \vee(X_2,S) > t, X_1 > \tau, X_2 > \tau) \\
&= P(X_1 \le \tau, X_2 > t) - P(X_1 \le \tau, X_2 > t, X_1 \le t, S \le t) \\
&+ P(X_1 > t, X_2 > \tau) - P(X_1 > t, X_2 > \tau, X_2 \le t, S \le t) \\
&= P(X_1 \le \tau, X_2 > t) - P(X_1 \le \tau, X_2 > t, S \le t) \\
&+ P(X_1 > t, X_2 > \tau) - P(X_1 > t, \tau < X_2 \le t, S \le t) \\
&= P(X_1 \le \tau, X_2 > t, S > t) + P(X_1 > t, X_2 > \tau) \\
&\quad -P(X_1 > t, X_2 > \tau, S \le t) + P(X_1 > t, X_2 > t, S \le t) \\
&= P(X_1 \le \tau, X_2 > t, S > t) + P(X_1 > t, X_2 > \tau) \\
&\quad -P(X_1 > t, X_2 > \tau, S \le t) + P(X_1 > t, X_2 > t, S \le t) \\
&= P(X_2 > t, S > t) - P(X_1 > \tau, X_2 > t, S > t) \\
&\quad +P(X_1 > t, X_2 > \tau, S > t) + P(X_1 > t, X_2 > t) - P(X_1 > t, X_2 > t, S > t) \\
&= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,X_2,S}(\tau,t,t) + \bar{F}_{X_1,X_2,S}(t,\tau,t) + \bar{F}_{X_1,X_2}(t,t) \\
&\quad -\bar{F}_{X_1,X_2,S}(t,t,t). \qquad (6)
\end{aligned}
$$

From Equations (5) and (6), the reliability functions of the system lifetimes $T_1^{[0,\tau]}$ and $T_2^{[0,\tau]}$ are $\overline{F}_{T_1^{[0,\tau]}}(t) = g_1(t)I_{[0,\tau)}(t) + g_2(t)I_{[\tau,\infty)}(t)$, and $\overline{F}_{T_2^{[0,\tau]}}(t) = z_1(t)I_{[0,\tau)}(t) + z_2(t)I_{[\tau,\infty)}(t)$ where $g_i(t)$ and $z_i(t)(i = 1,2)$ are defined by

$$
g_1(t) = \bar{F}_{X_2}(t) - F_{X_1,S}(t,t) + F_{X_1,X_2,S}(t,t,t), \qquad (7)
$$

$$
g_2(t) = \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,X_2,S}(\tau,t,t) + \bar{F}_{X_1,X_2,S}(t,\tau,t) + \bar{F}_{X_1,X_2}(t,t) - \bar{F}_{X_1,X_2,S}(t,t,t), \qquad (8)
$$

$$z_1(t) = \bar{F}_{X_1}(t) - F_{X_2,S}(t,t) + F_{X_1,X_2,S}(t,t,t), \quad (9)$$

$$z_2(t) = \bar{F}_{X_1,S}(t,t) - \bar{F}_{X_1,X_2,S}(t,\tau,t) + \bar{F}_{X_1,X_2,S}(\tau,t,t) + \bar{F}_{X_1,X_2}(t,t) - \bar{F}_{X_1,X_2,S}(t,t,t). \quad (10)$$

Remark 2.1. Notice that $\lim_{t\to\tau^-} g_1(t) = \lim_{t\to\tau^+} g_2(t)$ and $\lim_{t\to\tau^-} z_1(t) = \lim_{t\to\tau^+} z_2(t)$. Therefore, the reliability functions of the system lifetimes $T_1^{[0,\tau]}$ and $T_2^{[0,\tau]}$ are continuous in t for all $t > 0$.

3 Main results

System lifetimes are random variables and then partially orders should be considered for comparison purposes. Among various partially orders, stochastic orders are commonly used in reliability analyses. In this section, the main result of this paper is presented which hold under a general setting for the component and spare lifetimes. In the rest of this paper and for lifetimes U_1, U_2 and U_3, let $\bar{F}_{U_1|(U_2,U_3)}(u_1|u_2,u_3) := P(U_1 > u_1 | U_2 > u_2, U_3 > u_3)$.

Proposition 3.1. *Let X_1, X_2 and S be dependent random variables and τ be a given positive constant. If $[X_1|S=s] \leq_{st} [X_2|S=s]$ for all $s \geq 0$ and $\bar{F}_{X_1|(X_2,S)}(\tau|t,t) \leq 1/2$ and $\bar{F}_{X_2|(X_1,S)}(\tau|t,t) \geq 1/2$ for $t > \tau$,then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$.*

Proof. [i] For $0 < t \leq \tau$, Equations (7) and (9) conclude

$$\begin{aligned}\overline{F}_{T_1^{[0,\tau]}}(t) - \overline{F}_{T_2^{[0,\tau]}}(t) &= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,S}(t,t) \\ &= \int_t^{+\infty} \Big(P(X_2 > t|S=s) - P(X_1 > t|S=s) \Big) dF_S(s) \geq 0, \end{aligned} \quad (11)$$

since $X_1|S=s \leq_{st} X_2|S=s$ for all s¿0. For $t > \tau$, Equations (8) and (10) imply

$$\begin{aligned}\overline{F}_{T_1^{[0,\tau]}}(t) - \overline{F}_{T_2^{[0,\tau]}}(t) &= \bar{F}_{X_2,S}(t,t) - \bar{F}_{X_1,S}(t,t) + 2\bar{F}_{X_1,X_2,S}(t,\tau,t) - 2\bar{F}_{X_1,X_2,S}(\tau,t,t) \\ &= \bar{F}_{X_2,S}(t,t)(1 - 2\bar{F}_{X_1|X_2,S}(\tau|t,t)) + \bar{F}_{X_1,S}(t,t)(1 - 2\bar{F}_{X_2|X_1,S}(\tau|t,t)) \\ &\geq 0. \end{aligned} \quad (12)$$

Since $\bar{F}_{X_1|(X_2,S)}(\tau|t,t) \leq 1/2$ then $\bar{F}_{X_2|(X_1,S)}(\tau|t,t) \geq 1/2$, and the desired result follows. □

To compare the system lifetimes, one can find some results under some conditions. In sequel, system lifetimes are compared for independent variabels.

4 Independent spare

Here, some conditions are assumed which simplify the main result in Proposition 3.1. First, assume that the spare is independent of original component lifetimes X_1 and X_2 while the components may be dependent. The next corollary is immediately followed from Proposition 3.1. More details of proof are given in Jeddi and Doostparast [10].

Corollary 4.1. *Let S be independent of (X_1, X_2). If $X_1 \leq_{st} X_2$ and for $t > \tau$, $\bar{F}_{X_1|X_2}(\tau|t) \leq 1/2$ and $\bar{F}_{X_2|X_1}(\tau|t) \geq 1/2$ then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$.*

Proposition 4.2. *Let X_1, X_2 and S be independent. If $X_1 \leq_{st} X_2$ and $m_1 < \tau < m_2$, where m_1 and m_2 stand for medians of X_1 and X_2, respectively. Then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$.*

Proof. [i] For $0 < t \leq \tau$, Equations (7) and (9) conclude

$$\begin{aligned}
\overline{F}_{T_1{}^{[0,\tau]}}(t) - \overline{F}_{T_2{}^{[0,\tau]}}(t) &= g_1(t) - z_1(t) \\
&= F_{X_1}(t) - F_{X_2}(t) + F_{X_2}(t)F_S(t) - F_{X_2}(t)F_S(t) \\
&= \bar{F}_S(t)(F_{X_1}(t) - F_{X_2}(t)) \geq 0, \qquad (13)
\end{aligned}$$

since $X_1 \leq_{st} X_2$. For $t > \tau$, Equations (8) and (10) imply

$$\begin{aligned}
\overline{F}_{T_1{}^{[0,\tau]}}(t) - \overline{F}_{T_2{}^{[0,\tau]}}(t) &= g_2(t) - z_2(t) \\
&= \bar{F}_{X_2}(t)\bar{F}_S(t) - \bar{F}_{X_1}(t)\bar{F}_S(t) \\
&\quad +2(\bar{F}_{X_1}(t)\bar{F}_{X_2}(\tau)\bar{F}_S(t) - \bar{F}_{X_1}(\tau)\bar{F}_{X_2}(t)\bar{F}_S(t)) \\
&= \bar{F}_{X_2}(t)\bar{F}_S(t)(1 - 2\bar{F}_{X_1}(\tau)) + \bar{F}_{X_1}(t)\bar{F}_S(t)(2\bar{F}_{X_2}(\tau) - 1) \\
&\geq 0, \qquad (14)
\end{aligned}$$

since $\tau > m_1$ and $\tau < m_2$. and the desired result follows. □

Proposition 4.2 says that if component and spare lifetimes are independent, the spare should allocate to the weaker component at least up to its median lifetime and then before reaching the median lifetime of the other component, the spare must switch.

Remark 4.3. The distribution of S in Proposition 4.2 is general and the conditions do not rely on the Df of S.

Example 4.4. Let X_1, X_2, S be independent exponential random variables with means $1/\lambda_1$, $1/\lambda_2$ and $1/\lambda_3$, respectively. If $\lambda_1 > \lambda_2$, then Proposition 4.1 implies that $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ provided that the switching occurs after $\frac{\ln 2}{\lambda_1}$ but before $\frac{\ln 2}{\lambda_2}$, that is $\frac{\ln 2}{\lambda_1} < \tau < \frac{\ln 2}{\lambda_2}$. □

Example 4.5. Let X_1, X_2 and S be independent and $X_i \sim Pa(\alpha_i, 1), i = 1,2$ where $\alpha_i > 0$ and $Pa(a,b)$ stands for the Pareto distribution type 1 with density $f(x) = \dfrac{a}{x^{a+1}}, x \geq 1$. It is easy to see that the medians of X_1 and X_2, respectively, are given by $m_1 = \sqrt[\alpha_1]{2}$ and $m_2 = \sqrt[\alpha_2]{2}$ are median of X_1 and X_2 respectively. If $\alpha_1 > \alpha_2$ and $\sqrt[\alpha_1]{2} < \tau < \sqrt[\alpha_2]{2}$ then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ from Proposition 4.2.

Example 4.6. Let φ be the class of absolutely continues distribution function F_θ of the form $F_\theta(x) = 1 - e^{-K_\theta(x)}, x > 0$, where $K_\theta(x)$ is increasing in x and positive function $\theta \in \Theta$. Then the probability of density function is given by $f_\theta(x) = k_\theta(x)e^{-K_\theta(x)}, x > 0$, where $k_\theta(x) = \dfrac{\partial}{\partial x}K_\theta(x)$. This class include several important distribution such as exponential, Pareto, Weibull and has been studied in literature; See e.g, Al-Hussaini [1] for more details. Let $X_1 \sim F_{\theta_1}(x)$ and $X_2 \sim F_{\theta_2}(x)$. Then medians X_1 and X_2 are $m_1 = K_{\theta_1}^{-1}(\ln 2)$ and $m_2 = K_{\theta_2}^{-1}(\ln 2)$ respectively. If $K_{\theta_1(x)} \geq K_{\theta_2(x)}$ and $K_{\theta_1}^{-1}(\ln 2) \leq \tau \leq K_{\theta_2}^{-1}(\ln 2)$, then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ from Proposition 4.2. For example let $K_\theta(x) = \left(\dfrac{x}{\lambda}\right)^\alpha$ and $\Theta = (\alpha, \lambda), \alpha, \lambda > 0$, that is X_i, $i = 1,2$, has the Weibull distribution with density function $f_{\alpha_i,\lambda}(x) = \alpha_i x^{\alpha_i - 1}\lambda^{\alpha_i} e^{-(x/\lambda)^{\alpha_i}}, \alpha_i, \lambda > 0$, therefore $m_i = K_{\theta_i}^{-1}(\ln 2) = \sqrt[\alpha_i]{\ln 2}$ where $\Theta_i = (\alpha_i, \lambda)$, $i = 1,2$. If $\alpha_1 > \alpha_2$ and $\sqrt[\alpha_1]{\ln 2} < \tau < \sqrt[\alpha_2]{\ln 2}$ then $T_1^{[0,\tau]} \geq_{st} T_2^{[0,\tau]}$ from Proposition 4.2. □

5 Conclusions

This paper derived the system reliability function consisting of two original components and an additional spare. The spare can switch among the original ones. The finding are under a general setting. The optimal scheme for switching was also provided. Some special cases which have also practical applications were also studied in details. The results of this paper may be extended in various directions. For example, one can study the system behaviour under some parametric conditions such as the multivariate distribution functions for the component lifetimes. Engineering systems including parallel-series and series-parallel as well as coherent systems are worth for consideration.

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A General Algorithm for Optimal Redundancy Allocation in Coherent Systems

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Abstract: The redundancy allocation to original system components is a common technique to improve the system reliability and its performance. This paper considers a coherent system consisting of n independent components equipped with m independent redundant components. Both common schemes for allocating the redundant components to the system, called *active* and *standby* redundancies are considered. The essential problem which is of great importance in redundancy allocation problem of systems is how to find the optimal allocation strategy such that the performance of the system be optimal in the sense of some stochastic orders. In view of the usual stochastic order and in order to maximize the system reliability, an algorithm is proposed to find the optimal redundancy allocation. We first introduce a new measure of component importance which is useful to find the best allocation for adding $m = 1$ redundant component and then use it to propose our algorithm when $m > 1$. This algorithm solves the optimal redundancy allocation problem and extends the most existing results in literature for particular structures such as series, parallel, k-out-of-n and those systems with $m = 1$ redundant component.

Keywords: Coherent systems, Importance measures, Optimal allocation, Redundancy, Stochastic orders.

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1 Introduction

Consider a system consisting of n components in which all components and the system are in working or failed state. The state of the system is completely determined by the states of the components. Let $\phi(x_1,\ldots,x_n)$ denote the state of the system and x_i denotes the state of the ith component for $i=1,\ldots,n(x_i=1$ means that the ith component is working and $x_i=0$ that it is not). ϕ is called the system structure function. The system is coherent if ϕ be increasing, that is, when the state of a component is improved, the state of the system can not be worse, and every component be relevant for the system, that is, ϕ is strictly increasing in each variable in at least a point. For details on coherent systems refer to [1].

The use of redundancy mechanisms is an important and effective way to improve the performance of the system. Two common schemes for allocating the redundant components to the system, are called *active* and *standby* redundancies. In the former, the redundant components are put in parallel to the original components of the system, while in later, they start working immediately after component failures. In fact the use of redundancy means that, the improvement of system performance via the improving of its components. It is therefore important to study the effect of improving system components in improvement of the whole system.

Suppose m is the number of redundant components and r_i is the number of redundant components added to the ith original system component. Then the vector $\mathbf{r}=(r_1,\ldots,r_n)$ where $\sum_1^n r_i=m$ is called an allocation policy. The relevant and essential problem related to the redundancy in systems is how to find the optimal allocation strategy such that the performance of the system be optimal in the sense of some stochastic orders. In this paper the usual stochastic order is considered and therefore we want to find the optimal allocation strategy such that the system reliability be maximized. Although the redundancy allocation to original system components is a common technique to improve system reliability but, allocation of redundant components is not an easy task and must be considered properly with respect to environmental working conditions and possible restrictions such as cost, volume and weight. Therefore, the problem of finding optimal allocations is important. In last three decades the redundancy allocation problem has been widely studied by many authors. The most existing studies are usually concerned to particular structures such as series, parallel and k-out-of-n systems or coherent systems with $m = 1$ redundant component.

It seems that this subject was considered first by [2]. They considered a series system with n independent and identical (iid) components having common distribution F and m iid active redundant components with common distribution G and showed that the strategy of balanced allocation, that is the allocation with $|r_i - r_j| \leq 1$, for all $i \neq j$ is optimal and maximizes the system reliability. Obviously the balanced allocation is not unique unless m/n be an integer. In this case the only balanced allocation is the equal allocation $\mathbf{r}^* = (m/n, \ldots, m/n)$. In view of the hazard rate order, [8] showed that the balanced allocation optimizes the failure rate of the system if $F = G$, that is the system failure rate is minimized. [7] obtained a stronger result and showed that if lnG/lnF is an increasing function then the balanced allocation is optimal and minimizes the failure rate of the system.

We recall that a random variable X with distribution function $F_X = 1 - \bar{F}_X$ is said to be less than Y in usual stochastic order and denoted by $X \leq_{st} Y$, if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all t. Also when X and Y are absolutely continuous random variables then X is said to be less than Y in hazard rate order and denoted by $X \leq_{hr} Y$, if $h_X(t) = f_X(t)/\bar{F}_X(t) \geq h_Y(t)$ for all t, where $\bar{F}$, f and h stand for survival, density and hazard(failure) rate functions, respectively. It is known that if $X \leq_{hr} Y$ then $X \leq_{st} Y$.

In view of some other stochastic orders [11]. considered the optimal allocation policy in a series system with $m = 1$ redundancy. ([5, 6]) considered the optimal allocation of redundancies in series and parallel systems with dependent components when $m \leq 2$. [10] considered k-out-of-n systems with independent components and m iid redundant components. ([3, 4]) considered the optimal allocation in active redundancy for some coherent systems containing symmetric components.

In a fixed point of time consider a coherent system with n components and structure ϕ and let

$$h(\mathbf{p}) = E[\phi(\mathbf{X})] = P(\phi(\mathbf{X}) = 1)$$

be the reliability function of the system where $\mathbf{X} = (X_1, \ldots, X_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$ and $p_i = P(X_i = 1)$ is the reliability of ith component.
In during of the time we denote the system lifetime by $T = \Phi(T_1, \ldots, T_n)$ where T_i is the lifetime of ith component. We assume that the system components are independent, that is X_i's or T_i's are independent random variables. Also assume that T_i's are nonnegative and absolutely continuous. The reliability function of the system at time t is denoted by

$$\bar{F}_T(t) = P(T > t) = h(\mathbf{p}_t) = h(p_1(t), \ldots, p_n(t))$$

where $p_i(t) = \bar{F}_i(t) = P(T_i > t)$.

The rest of this paper is organized as follows: In Section 2 we introduce a new measure of component importance which is an extension of the well known Birnbaum measure of importance and is useful in both active and standby redundancies problems in coherent systems with $m = 1$ redundant component. Using this an algorithm is proposed to find the optimal allocation when $m > 1$ in Sections 3.

2 A new measure of component importance useful in active and standby redundancies

In this section, we consider a coherent system with n independent components and study the effect of an redundant component on the system reliability and then give our new measure of component importance which is useful in redundancy problems of coherent systems, considered in the next section. For the sake of completeness we first give the effect of improving one component on the system reliability, a result obtained by [9].

Lemma 2.1. Let Δ_i denote the increase of system reliability due to the increasing of ith component reliability as much as δ_i. Then

$$\Delta_i = \delta_i I_B(i) \tag{1}$$

where

$$I_B(i) = P(\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X}) = 1) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}) = \frac{\partial h(\mathbf{p})}{\partial p_i}$$

is the well known Birnbaum importance measure of the ith component.

Regarding the above lemma we now consider three cases as follows.
Case 1. If $\delta_i = \delta$, $i = 1, \ldots, n$, that is the improvement of all components be the same, then from (2.1) we see that in view of the Birnbaum measure of importance, improvement of the most important component causes the largest increasing in system reliability. In other words the Birnbaum measure of importance is crucial to find the best component in order to increase the system reliability. This is not case if $\delta_i \neq \delta$ and therefore a new measure of importance is needed. One may use $\Delta_i = \delta_i I_B(i)$ as the new measure of importance for component i. But it is not applicable in redundancy allocation problem as Δ_i depends on the arbitrary value δ_i whereas in redundancy problem δ_i depends on the redundant component. It is explained in the following case.

Case 2. Suppose we want to allocate an active redundancy component with reliability p to a single system component. We assume that it is independent of all original system components. The question is how to find the optimal allocation. If we allocate it to the component i then p_i will be increased to $1-(1-p_i)(1-p) = 1-q_iq$ and therefore $\delta_i = 1-q_iq-p_i = q_i-q_iq = q_ip$. Hence $\Delta_i = q_ipI_B(i)$. Based on this we now introduce our new measure of importance for component i as follow

$$I_{\text{AR}}(i) = (1 - p_i)I_B(i). \tag{2}$$

AR in $I_{\text{AR}}(i)$ refers to active redundancy. It is a generalization of $I_B(i)$ as it depends to p_i the reliability of component i but $I_B(i)$ does not. Also note that in this case the active component is exposed to all system components under the same condition. Hence the optimal allocation is the component that has the largest $I_{\text{AR}}(.)$.

Case 3. In this case we want to allocate one independent standby component with reliability p and lifetime S to a single system component and find the optimal allocation. If we allocate it to the component i with lifetime T_i then p_i will be increased to $p_i * p$. By $p_i * p$ we mean $\bar{F}_i * \bar{F}(t) = P(T_i + S > t)$, the convolution of $\bar{F}_i$ and $\bar{F}$, which are the reliability functions of T_i and S, respectively. Therefore

$$\delta_i = p_i * p - p_i = P(T_i + S > t) - P(T_i > t)$$

and our new measure of importance for component i is

$$I_{\text{SR}}(i) = (p_i * p - p_i)I_B(i). \tag{3}$$

SR in $I_{\text{SR}}(i)$ refers to standby redundancy. Hence in this case the optimal allocation is the component that has the largest $I_{\text{SR}}(.)$.

Remark 2.2. If the system components are identical, that is $p_1 = \cdots = p_n$ we then in both cases 2 and 3 have, $\delta_1 = \cdots = \delta_n$ and therefore in order to find the optimal allocations, $I_{\text{AR}}(i)$ and $I_{\text{SR}}(i)$ will be equivalently reduced to $I_B(i)$. Also note that in case 3 $I_{\text{SR}}(i)$ is dependent to the lifetime distributions of original and spare components whereas in case 2 this is not case for $I_{\text{AR}}(i)$. To obtain $I_B(i)$ in case 3, it is sufficient to replace p_i in case 2 by $p_i(t) = P(T_i > t) = \bar{F}_i(t)$.

3 An algorithm for optimal Redundancy allocation

In this section we consider the active and standby redundancy problems in a coherent system consisting of n independent components equipped with m independent redundant components. Assume that the original and spare components are also independent. We want to find the optimal allocation in the sense of usual stochastic order.

We also assume that the functional form of the system reliability function $h(\mathbf{p}) = h(p_1, \ldots, p_n)$, is known. Without loss of generality we assume that $p'_1 \geq \cdots \geq p'_m$ where p'_j is the reliability of the jth redundant component. Note that if we want to add the redundant component j to the original component i in fact $\Delta_i = p'_j(1-p_i)I_B(i)$ has the largest value and since $p'_1 \geq \cdots \geq p'_m$ therefore the redundant components should be added to the system components consecutively, first the redundant component 1 and then redundant component 2, etc. Therefore it is enough in each stage, that we first find the original component with largest value of $I_{\text{AR}}(i) = (1-p_i)I_B(i)$ and then add the current redundant component to it. In other words the measure of $I_{\text{AR}}(i) = (1-p_i)I_B(i)$ is crucial to find the optimal allocation.

Now based on the measure of $I_{\text{AR}}(i)$ given in section 2 we propose the following algorithm to find the optimal allocation in case of active redundancy.

An algorithm for optimal allocation in active redundancy

Input: $p_1, \ldots, p_n$, m, $p'_1 \geq \cdots \geq p'_m$ and $h(p_1, \ldots, p_n)$.

Output: The optimal allocation $\mathbf{r}^* = (r_1, \ldots, r_n)$.

Step 0. Put $I = 1$ and $r_i = 0$ and $q'_j = 1 - p'_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

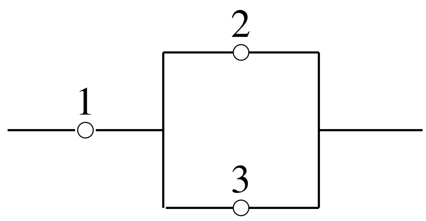
Step 1. Compute $I_B(i) = \frac{\partial h(\mathbf{p})}{\partial p_i}$, $q_i = 1 - p_i$, and $I_{\text{AR}}(i) = q_i I_B(i)$ for $i = 1, \ldots, n$.

Step 2. Determine the i^* such that

$$I_{\text{AR}}(i^*) = max\{I_{\text{AR}}(i), i = 1, \ldots, n\}.$$

Step 3. Put $r_{i^*} = r_{i^*} + 1$ and $p_{i^*} = 1 - q'_I q_{i^*}$ and update the system reliability

function $h(p_1,\ldots,p_n)$.
Step 4. If $I=m$, Stop. Otherwise put $I=I+1$ and Goto Step 1.
To illustrate how the above algorithm works see the following example.
Example. Consider the following series-parallel system.



Suppose $p_1=0.4$, $p_2=0.8$ and $p_3=0.3$ Also let $m=4$ and for simplicity suppose $p'_1=\cdots=p'_4=0.5$. We follow the above algorithm step by step to find the optimal allocation such that the reliability of the system be maximized. It is known that for this system we have $h(p_1,p_2,p_3)=p_1p_2+p_1p_3-p_1p_2p_3=0.404$. Therefore $I_B(1)=p_2+p_3-p_2p_3=0.86$, $I_B(2)=p_1-p_1p_3=0.28$ and $I_B(3)=p_1-p_1p_2=0.08$. Also $I_{\text{AR}}(1)=q_1I_B(1)=0.516$ and similarly $I_{\text{AR}}(2)=0.056=I_{\text{AR}}(3)$. We get $i^*=1$. That is the first redundant component should be added to component 1. Hence $r_1=1$ and $p_1=1-0.5\times0.6=0.7$. Using this new value of p_1 we update $h(p_1,p_2,p_3)$ to allocate the second redundant. We have $I_B(1)=0.86$, $I_B(2)=0.49$ and $I_B(3)=0.14$. Also $I_{\text{AR}}(1)=q_1I_B(1)=0.258$, $I_{\text{AR}}(2)=0.098=I_{\text{AR}}(3)$. We again have $i^*=1$ and therefore $r_1=2$ and $p_1=1-qp_1=1-0.5\times0.3=0.85$. That is the second redundant component should be again added to component 1. Similarly in third repetition we obtain $i^*=1$, $r_1=3$ and $p_1=1-0.5\times0.15=0.925$ and finally in last repetition have $I_{\text{AR}}(1)=0.0645$, $I_{\text{AR}}(2)=I_{\text{AR}}(3)=0.1295$. Therefore $i^*=2$ or 3. That is two allocations $\mathbf{r}_1^*=(3,1,0)$ and $\mathbf{r}_2^*=(3,0,1)$ are both optimal. We note that although the optimal allocation is not unique but one can simply verify that both of $\mathbf{r}_1^*$ and $\mathbf{r}_2^*$ lead to the unique maximum value of $h(p_1,p_2,p_3)$ which is equal to 0.86025. This holds true in general.

Remark. One can simply obtain an algorithm to find the optimal allocation in case of standby redundancy if $I_{\text{AR}}(i)$ be replaced by $I_{\text{SR}}(i)$ in the above algorithm. As mentioned before we note that $I_{\mathrm{SR}}(i)$ is a time dependent measure and the lifetime distributions of original and redundant components are needed.

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Discrete Multicomponent Stress-strength Inference in Proportional Hazard Model

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Abstract: In stress-strength models, consider a system which has k independent strength components and each component is constructed by a pair of dependent elements. These elements $(X_1,Y_1),(X_2,Y_2),\ldots,(X_k,Y_k)$ follow a discrete bivariate proportional hazard rate family and each element is exposed to a common random stress T which follows a discrete univariate proportional hazard rate family. The system is regarded as operating only if at least s out of k $(1 \leq s \leq k)$ strength variables exceed the random stress. In this paper, based on a general form of discrete lifetime distribution in proportional hazard rate models, the estimation of multicomponent stress-strength reliability parameter is studied. Finally, as an example the model have studied in a new bivariate Gemometric distribution.

Keywords: Discrete proportional hazard rate model (DPHM), Maximum likelihood estimator (MLE), Method of proportion (MP), Reliability, Stress-strength model, Telescopic representation.

1 Introduction

Stress-strength models are one of the most important issues with many citations in reliability and engineering. In the reliability context, the stress-strength model can be described as an assessment of reliability of a system

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in terms of random variables X representing stress experienced by the system and Y representing the strength of the system available to overcome the stress. If the stress exceeds the strength, then the system will fail. Thus $R = P(X < Y)$ is the reliability considering the failure mode described by the stress-strength relationship. This main idea was introduced by [3] and developed by [4]. Estimation of $R = P(X < Y)$ when the random variables X and Y follow a specified distribution has been extensively discussed by many authors in the literature. For a detailed survey on point and interval estimation of stress-strength models using different approaches one can refer [11]. [6] proposed some new approximate inferential methods for the reliability estimation in the stress-strength model when the stress and strength variables are independent normal random variables with unknown means and variances. [10] studied the reliability estimation of the stress-strength model when strength variables followed finite mixture of two parameter Lindley distribution and stress variables followed exponential distribution, Lindley distribution and finite mixture of two parameter Lindley distribution, respectively. [12] derived the estimation of $R = P(Y < X)$ when X and Y followed three-parameter generalized Rayleigh distributions with the same scale and location parameters but different shape parameters.

The above mentioned system is only a single-component system, but this assumption is not enough to cope with more cases. In fact, with the development of the science technology and manufactory technique, there are many multicomponent systems appearing in our daily life, such as the mouse, keyboards, IT hardware, aero-engines, and so on. It is significantly meaningful to study the reliability of multicomponent system in stress-strength models. Reliability analysis for a multicomponent survival stress-strength model based on exponential distributions was studied by [13]. [5] got some conclusions about stress-strength reliability under multi-state systems modeling, and he also studied the multicomponent form. [16, 17] discussed the reliability of multicomponent stress-strength model based on generalized exponential distribution, Burr-XII distribution and two parameter exponentiated Weibull distribution. An n-component-standby system stress-strength model was analyzed by [9].

Most of the works in estimation of R is based on continuous distributions for X and Y. Estimation of R based on discrete or categorical data was also addressed by some authors. [14] considered the estimation of R with geometric distribution for stress and strength random variables. [18] obtained MLE and UMVUE of R using negative binomial distribution. [15] obtained the Bayes estimator of R when X and Y have independent two-parameter geometric distribution. [1] obtained the estimator of R when the stress and strength components are independent Poisson random variables. [7] studied the estimation of stress-strength reliability using discrete phase type distribution.

In this study, we combine the multicomponent stress-strength model with a discrete bivariate distribution. We consider a system which has k statistically independent and identically distributed strength components, each of which consists of a series system of two statistically dependent elements exposed to a common stress. The system functions if $s(1 \leq s \leq k)$ or more components simultaneously operate.

2 Model description

Using the method of [2] for generating bivariate distributions, we introduced a general method for generating the bivariate discrete distributions.

Let X be any non-negative discrete integer valued random variable, then the Telescopic form (denoted by $X \sim T(q, k_\theta)$) of its pmf, cdf and reliability function ($R_X(x) = P(X \geq x)$) for $x \in D = \{0, 1, 2, \ldots\}$ are as follow respectively (Rezaei et al. 2000)

$$p_X(x) \;=\; q^{k_\theta(x)} - q^{k_\theta(x+1)}, \tag{1}$$

$$\begin{aligned} F_X(x) &= 1 - q^{k_\theta(x+1)}, \\ R_X(x) &= q^{k_\theta(x)}, \end{aligned}$$

where $0 < q < 1$ and θ is a parameter vector (which may contains q) and $k_\theta(x) = \frac{\ln \overline{F}(x-1)}{\ln q}$ is an increasing function of $x \in D$ with $k_\theta(0) = 0$ and $k_\theta(\infty) = \infty$.

Table 1 shows the form of $k_\theta(x)$ for some discrete distributions.

Table 1: *Some discrete distributions in telescopic forms*

Distribution of X	$k_\theta(x)$	PMF of X
Geometric	x	$q^x(1-q)$
Discrete Weibull	x^θ	$q^{x^\theta} - q^{(x+1)^\theta}$
Discrete Rayleigh	$\frac{x^2}{2}$	$q^{\frac{x^2}{2}} - q^{\frac{(x+1)^2}{2}}$
Discrete Gompertz	$e^{\theta x} - 1$	$q^{e^{\theta x}-1} - q^{e^{\theta(x+1)}-1}$
Brittle-Fracture	$x^{2r}e^{-\frac{\beta}{x^2}}$	$q^{x^{2r}e^{-\frac{\beta}{x^2}}} - q^{(x+1)^{2r}e^{-\frac{\beta}{(x+1)^2}}}$
Linear-Exponential	$x + \frac{\beta}{2\alpha}x^2$	$q^{x+\frac{\beta}{2\alpha}x^2} - q^{(x+1)+\frac{\beta}{2\alpha}(x+1)^2}$
Discrete Burr XII	$\ln(1+x^\alpha)$	$q^{\ln(1+x^\alpha)} - q^{\ln(1+(x+1)^\alpha)}$
Discrete Kumaraswamy	$-\ln(1-x^\beta)$	$q^{-\ln(1-x^\beta)} - q^{-\ln(1-(x+1)^\beta)}$

Now, let X_1, X_2 and X_3 be independent random variables with $X_i \sim T(q_i, k_\theta)$, where $q_i \in (0,1)$, for $i = 1,2,3$. Let

$$Z_1 = \min(X_1, X_3) \quad \text{and} \quad Z_2 = \min(X_2, X_3).$$

Hence, we define the random bivariate vector (Z_1, Z_2). We use the notation $(Z_1, Z_2) \sim BVT(q_1, q_2, q_3, k_\theta)$. The reliability function and joint pmf of (Z_1, Z_2), are given by

$$\begin{aligned} R_{Z_1,Z_2}(z_1,z_2) &= P(Z_1 \geq z_1, Z_2 \geq z_2) = P(X_1 \geq z_1, X_2 \geq z_2, X_3 \geq w) \\ &= R_{X_1}(z_1)R_{X_2}(z_2)R_{X_3}(w) \\ &= q_1^{k_\theta(z_1)} q_2^{k_\theta(z_2)} q_3^{k_\theta(w)}, \end{aligned}$$

$$\begin{aligned} P(Z_1 = z_1, Z_2 = z_2) &= R_{Z_1,Z_2}(z_1,z_2) - R_{Z_1,Z_2}(z_1^+,z_2) - R_{Z_1,Z_2}(z_1,z_2^+) + R_{Z_1,Z_2}(z_1^+,z_2^+) \\ &= q_1^{k_\theta(z_1)} q_2^{k_\theta(z_2)} q_3^{k_\theta(w)} - q_1^{k_\theta(z_1+1)} q_2^{k_\theta(z_2)} q_3^{k_\theta(w')} \end{aligned}$$

$$- \quad q_1^{k_\theta(z_1)} q_2^{k_\theta(z_2+1)} q_3^{k_\theta(w'')} + q_1^{k_\theta(z_1+1)} q_2^{k_\theta(z_2+1)} q_3^{k_\theta(w''')}$$

where $w = \max(z_1, z_2)$, $w' = \max(z_1+1, z_2)$, $w'' = \max(z_1, z_2+1)$ and $w''' = \max(z_1+1, z_2+1)$ for $(z_1, z_2) \in D^2$.

It is easy to show that if $(X,Y) \sim BVT(q_1, q_2, q_3, k_\theta)$, then the marginal distributions are distributed as $X \sim T(q_1q_3, k_\theta)$ and $Y \sim T(q_2q_3, k_\theta)$. Moreover, the distribution of $\min(X,Y)$ is $T(q_1q_2q_3, k_\theta)$.

Notice that the random variables X and Y become statistically independent as $q_3 \longrightarrow 1^-$ and the correlation between X and Y increases as q_3 decreases. Hence, q_3 can be regarded as a correlation control parameter.

We consider a system which has k statistically independent and identically distributed strength components and each component is constructed by a pair of statistically dependent elements. The system is subjected to a common stress and works if at least s $(1 \leq s \leq k)$ components simultaneously operate; and a component is alive only if the weakest elements is operating.

We assume the strength vectors are distributed as $(X_i, Y_i) \sim BVT(q_1, q_2, q_3, k_\theta)$ and a common stress variable has distributed as $T \sim T(p, k_\theta)$. Let $Z_i = \min(X_i, Y_i)$, then $Z_i \sim T(q, k_\theta)$, $i = 1, \ldots, k$ where $q = q_1q_2q_3$. In terms of these random variables, the system is working if at least s $(1 \leq s \leq k)$ of the Z_i strength variables operate when the common stress variable T is carried out.

Let $T, Z_1, \ldots, Z_k$ be statistically independent, $g_T(t)$ be the pmf of T and $R_Z(z)$ be the common reliability function of $Z_1, \ldots, Z_k$. The reliability in a multicomponent stress-strength model is given by

$$\begin{aligned} R_{s,k} &= P(\text{at least} s \text{ of the} (Z_1, \ldots, Z_k) \text{is more or equal } T) \\ &= \sum_{i=s}^{k} \sum_{t=0}^{\infty} \binom{k}{i} P(Z \geq t)^i P(Z < t)^{k-i} P(T = t) \\ &= \sum_{t=0}^{\infty} g_T(t) \sum_{i=s}^{k} \binom{k}{i} R_Z(t)^i (1 - R_Z(t))^{k-i} \\ &= E_T(P(B_t \geq s)) = E_T(R_{B_T}(s)), \end{aligned}$$

where $B_t \sim Binomial(k, R_Z(t))$. In our case, where $(X_i, Y_i) \sim BVT(q_1, q_2, q_3, k_\theta)$ and $T \sim T(p, k_\theta)$, $R_{s,k}$ is given by

$$R_{s,k} = \sum_{i=s}^{k}\sum_{t=0}^{\infty}\binom{k}{i} q^{ik_\theta(t)}(1-q^{k_\theta(t)})^{k-i}(p^{k_\theta(t)} - p^{k_\theta(t+1)}), \qquad (2)$$

where $q = q_1 q_2 q_3$.

In another scenario, we consider a system which has k statistically independent and identically distributed strength components and each component is constructed by two random variables Y_1 and Y_2 which are satisfied the discrete proportional hazard rate model (DPHM) with resilience parameter $\beta > 0$. i.e. for $i = 1,2$; $R_{Y_i}(t) = [R_0(t)]^\beta$ for $t = 0,1,\ldots$, where $R_0(t)$ is the reliability function of the baseline distribution. Therefore the reliability function of the random variable $Z = \min(Y_1, Y_2)$ is $R_Z(t) = [R_0(t)]^{2\beta}$. So, if the stress variable, T is also stisfied the DPHM with resilience parameter $\alpha > 0$ (i.e. $R_T(t) = [R_0(t)]^\alpha$), we have,

$$\begin{aligned} R_{s,k} &= P(\text{at least}\, s \text{ of the}\, (Z_1,\ldots,Z_k)\, \text{is more or equal}\; T) \\ &= \sum_{i=s}^{k}\sum_{t=0}^{\infty}\binom{k}{i}[R_0(t)]^{2\beta i}(1-[R_0(t)]^{2\beta})^{k-i}(R_0(t)^\alpha - R_0(t+1)^\alpha) \end{aligned}$$

3 Estimations of the R

In this section two estimation for R is presented.

3.1 MLE of R

To obtain the MLE of $R_{s,k}$, suppose that n systems are put on experiment and we have the following potential data $(X_{i1}, Y_{i1}), (X_{i2}, Y_{i2}), \ldots, (X_{ik}, Y_{ik})$ and T_i, $i = 1,\ldots,n$, but the actual observed data are $Z_{i1}, Z_{i2}, \ldots, Z_{ik}$ and T_i , $i = 1,\ldots,n$. Suppose that $(X_i, Y_i) \sim BVT(q_1, q_2, q_3, k_\theta)$, $Z_i = \min(X_i, Y_i) \sim T(q, k_\theta)$; where $q = q_1 q_2 q_3$ and $T \sim T(p, k_\theta)$, Then, the likelihood function of these observed samples is given as

$$\begin{aligned} L(q,p|k_\theta,\underline{Z},\underline{t}) &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k} p_Z(z_{ij})\right) p_T(t_i) \\ &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k}(R_Z(z_{ij})-R_Z(z_{ij}+1))\right)(R_T(t_i)-R_T(t_i+1)) \end{aligned}$$

and in DPHM in which, $R_Z(u)=[R_0(u)]^{2\beta}$, $R_T(u)=[R_0(u)]^{\alpha}$ and $R_0(u)=q'^{k_\theta(u)}$, where q' and θ are known parameters, we have,

$$\begin{aligned} L(\alpha,\beta|k_\theta,\underline{Z},\underline{t}) &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k}(R_0(z_{ij})^{2\beta}-R_0(z_{ij}+1)^{2\beta})\right)(R_0(t_i)^{\alpha}-R_0(t_i+1)^{\alpha}) \\ &= \prod_{i=1}^{n}\left(\prod_{j=1}^{k} q'^{2\beta k_\theta(z_{ij})}-q'^{2\beta k_\theta(z_{ij}+1)}\right)(q'^{\alpha k_\theta(t_i)}-q'^{\alpha k_\theta(t_i+1)}) \end{aligned}$$

and the log-likelihood function is

$$l(\alpha,\beta|k_\theta,\underline{Z},\underline{t}) = \sum_{i=1}^{n}\sum_{j=1}^{k}\ln(q'^{2\beta k_\theta(z_{ij})}-q'^{2\beta k_\theta(z_{ij}+1)})+\sum_{i=1}^{n}\ln(q'^{\alpha k_\theta(t_i)}-q'^{\alpha k_\theta(t_i+1)}).$$

So, the MLE of α and β can be obtained by solution of the following equations,

$$\begin{aligned} \sum_{i=1}^{n}\frac{k_\theta(t_i)q'^{\alpha k_\theta(t_i)}}{p_T(t_i)} &= \sum_{i=1}^{n}\frac{k_\theta(t_i+1)q'^{\alpha k_\theta(t_i+1)}}{p_T(t_i)} \\ \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{k_\theta(z_{ij})q'^{2\beta k_\theta(z_{ij})}}{p_Z(z_{ij})} &= \sum_{i=1}^{n}\sum_{j=1}^{k}\frac{k_\theta(z_{ij}+1)q'^{2\beta k_\theta(z_{ij}+1)}}{p_Z(z_{ij})} \end{aligned}$$

3.2 Method of Proportions

In discrete distributions [8] have proposed a method called "*method of proportions*" for estimation the parameters. Based on this method in DPHM we have,

$$p_Z(0)=p(Z=0)=1-q'^{2\beta k_\theta(1)}. \tag{3}$$

Therefore if N_0 be the number of zero's in the sample of $z_1,\ldots,z_n$, then the proportion $\frac{N_0}{n}$ estimates the probability $p_Z(0)$, so $\hat{\beta}_{mp}$ as the estimator of

β, can be expressed in a simple form of

$$\hat{\beta}_{mp} = \frac{1}{2k_\theta(1)} \log_{q'}(1 - \frac{N_0}{n}), \tag{4}$$

and in similar way the estimator of α, $\hat{\alpha}_{mp}$, can be obtained as follow,

$$\hat{\alpha}_{mp} = \frac{1}{k_\theta(1)} \log_{q'}(1 - \frac{N'_0}{n}), \tag{5}$$

where N'_0 is the number of zero's in the sample of t_i; $i = 1, \ldots, n$.

4 Results for Geometric distribution

First of all, we define a new three-paramtric bivariate Geometric distribution as follow.

Definition 4.1. The (X,Y) has three-parameter bivariate Geometric distribution and denoted by $(X,Y) \sim BGeo(q_1,q_2,q_3)$, if it has the following reliability and joint pmf as follow,

$$R_{X,Y}(x,y) = q_1^x q_2^y q_3^{\max(x,y)} = \begin{cases} (q_1q_3)^x q_2^y & x \geq y \\ q_1^x (q_2q_3)^y & x < y \end{cases}$$

$$p_{X,Y}(x,y) = \begin{cases} (q_1^x - q_1^{x+1})\left((q_2q_3)^y - (q_2q_3)^{y+1}\right) & x < y \\ (q_1q_3)^x q_2^y - (q_1q_3)^{x+1} q_2^y - (q_2q_3)^{y+1} q_1^x + (q_1q_3)^{x+1} q_2^{y+1} & x = y \\ (q_2^y - q_2^{y+1})\left((q_1q_3)^x - (q_1q_3)^{x+1}\right) & x > y \end{cases}$$

Therefore, using (2), $R_{s,k}$ is as follow,

$$R_{s,k} = \sum_{i=s}^{k} \sum_{t=0}^{\infty} \binom{k}{i} q^{it}(1-q^t)^{k-i}(p^t - p^{t+1}),$$

where $q = q_1q_2q_3$, and in DPHM, we have

$$R_{s,k} = \sum_{i=s}^{k} \sum_{t=0}^{\infty} \binom{k}{i} q'^{2\beta ti}(1-q'^{2\beta t})^{k-i}(q'^{\alpha t} - q'^{\alpha(t+1)})$$

Also, the MLE of α and β can be obtianed by the following equations,

$$\sum_{i=1}^{n} \frac{t_i q'^{\alpha t_i}}{p_T(t_i)} = \frac{q'}{1-q'} \sum_{i=1}^{n} \frac{q'^{\alpha t_i}}{p_T(t_i)}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{t_i q'^{\alpha t_i}}{p_T(t_i)} = \frac{q'}{1-q'}\sum_{i=1}^{n}\frac{q'^{\alpha t_i}}{p_T(t_i)}$$

Future of research

Among the topics that can be done in the future of this research, we can mention the comparison of MLE and MP estimators as well as the calculation of Bayesian estimators and confidence interval.

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E-Bayesian estimation and its E-Posterior risk for the reliability function derived from the exponential distribution

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Abstract: In this paper, we deal with E-Bayesian estimation and its E-Posterior risk for the reliability function derived from the exponential distribution under the scaled squared error loss function. The E-posterior risk (expected posterior risk) is used to measure the estimated error based on the E-Bayesian estimation. E-Bayesian estimators and formulas of E-posterior risk are derived. A Monte Carlo simulation is performed for comparison of the proposed estimators.

Keywords: E-Bayesian estimation, E-Posterior risk, Exponential distribution, Reliability function.

1 Introduction

The exponential distribution is an important distribution in the field of life testing and reliability theory. Consider that X denotes the time-to-failure of a specific device which has an exponential Exp(θ) distribution with probability density function (p.d.f.) given by

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0, \;\; \theta > 0, \tag{1}$$

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where θ represents the failure rate. The reliability (survival function) is given by

$$R(t) = P(X > t) = e^{-\theta t}. \quad (2)$$

Suppose that n items from an exponential distribution are placed on life test and a sample $\mathbf{X} = (X_1, \ldots, X_n)$ is obtained. Taking $S = \sum_{i=1}^{n} X_i$, the maximum likelihood estimator (MLE) of $R = R(t)$ based on $\mathbf{X}$ is given by

$$\hat{R}_{ML} = e^{-\frac{nt}{S}}.$$

Bayesian approachs in statistical problems require defining a prior distribution over the parameter space and loss function. Many Bayesian believe that just one prior can be elicited. In practice, the prior knowledge is vague and any elicited prior distribution is only an approximation to the true one. So, we elect to restrict attention to a given flexible family of priors. Various solutions to this problem have been proposed. One of the proposed solution is E-Bayesian approach, which was first introduced by [1]. The E-Bayesian estimator of the unknown parameter is the expectation of the Bayesian estimation on the basis of distribution of the hyperparameter(s), for more details, see ([2, 3, 4]), [6] and [7]. The E-Posterior risk of E-Bayesian estimation is the the expectation of the E-Bayesian estimation over the hyperparameter(s) which is proposed by [5] for the reliability function in a binomial distribution.

In this paper, we consider the problem of E-Bayesian estimation of the reliability function and its E-posterior risk derived from the exponential distribution using the scaled squared error loss (SSEL) function

$$L(R, \hat{R}) = \frac{(R - \hat{R})^2}{R^k},\ k = 0, 1, 2, \quad (3)$$

where $\hat{R}$ is an estimator of R. In section 2, we find the Bayesian estimators for the reliability function under the loss function (3). In section 3, we obtain the E-Bayesian estimators for the reliability function using a prior distribution of hyperparameter. The formulas of E-Posterior risk are derived in section 4. A Monte Carlo simulation is used for comparison of E-Bayesian estimators of the reliability function in section 5.

2 Bayesian estimators of reliability function

If $\mathbf{x} = (x_1, x_2, ..., x_n)$ be the sample observations from the exponential distribution (1), then the likelihood function is given by

$$L(\theta|\mathbf{x}) = \Pi_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta S}.$$

If we take conjugate prior of θ, namely $Exp(b)$, with p.d.f.

$$\pi(\theta|b) = be^{-b\theta}, \quad \theta > 0, \ \ b > 0. \tag{4}$$

Then, the posterior distribution of θ given $\mathbf{x}$ is $Gamma(n+1, S+b)$ with p.d.f.

$$\pi(\theta|\mathbf{x}) = \frac{(S+b)^{n+1}}{\Gamma(n+1)}\, \theta^n e^{-(S+b)\theta}, \ \ \theta > 0. \tag{5}$$

In the following Lemma, we present the Bayesian estimation of R under the loss function (3).

Lemma 2.1. *(i) Under the SSEL function (3), The Bayesian estimate of R based on observation $\boldsymbol{x}$ is given by*

$$\hat{R}^{Bk}(\boldsymbol{x}) = \frac{E[R^{1-k} \mid \boldsymbol{x}]}{E[R^{-k} \mid \boldsymbol{x}]}. \tag{6}$$

(ii) Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ be the sample observations from the exponential distribution given in (1), the prior distribution of θ is $Exp(b)$ given in (4) and $s = \sum_{i=1}^{n} x_i$. Then, the Bayesian estimation of R is given by

$$\hat{R}^{Bk}(\boldsymbol{x}) = (\frac{S+b-tk}{S+b+t-tk})^{n+1}, \ \ k = 0, 1, 2. \tag{7}$$

Proof. (i) The posterior risk of $\hat{R}$ based on observations $\mathbf{x} = (x_1, ..., x_n)$ can be expressed as

$$\rho(\pi, \hat{R}) \ \ = \ \ E[L(R, \hat{R})|\mathbf{x}]$$

$$
\begin{aligned}
&= E[\frac{(R-\hat{R})^2}{R^k}|\mathbf{x}] \\
&= E[R^{2-k}|\mathbf{x}] - 2\hat{R}E[R^{1-k}|\mathbf{x}] + \hat{R}^2 E[R^{-k}|\mathbf{x}]. \qquad (8)
\end{aligned}
$$

The Bayes estimate of R based on observation $\mathbf{x}$ is any estimate $\hat{R}^{Bk}(\mathbf{x})$ that minimizes the posterior risk (8) w.r.t. $\hat{R}$, which is given by

$$
\hat{R}^{Bk}(\mathbf{x}) = \frac{E[R^{1-k} \mid \mathbf{x}]}{E[R^{-k} \mid \mathbf{x}]}. \qquad (9)
$$

(ii) Using the posterior density given in (5), we obtain

$$
E[R^{1-k}|\mathbf{x}] = E[e^{-\theta t(1-k)}|\mathbf{x}] = (\frac{S+b}{S+b+t-tk})^{n+1}, \qquad (10)
$$

and

$$
E[R^{-k}|\mathbf{x}] = E[e^{\theta tk}|\mathbf{x}] = (\frac{S+b}{S+b-tk})^{n+1}. \qquad (11)
$$

Therefore, the Bayesian estimator of R under the loss function (3) is given by

$$
\hat{R}^{Bk}(\mathbf{x}) = \frac{E[R^{1-k}|\mathbf{x}]}{E[R^{-k}|\mathbf{x}]} = (\frac{S+b-tk}{S+b+t-tk})^{n+1}. \qquad (12)
$$

□

Remark 2.2. It can be shown that the following sequential relationship hold between the Bayesian estimators of R

$$
\hat{R}^{B2}(\mathbf{x}) < \hat{R}^{B1}(\mathbf{x}) < \hat{R}^{B0}(\mathbf{x})
$$

3 E-Bayesian estimation

Information on the appropriate prior is often inadequate to unambiguously specify a prior distribution. The problem of expressing uncertainty regarding prior information can be solved by using a class of prior distributions.

E-Bayesian inference deals with such a problem by constructing methods which are stable to such a lack of information.

Consider a prior $\pi(\theta|b)$ for θ with hyperparameter b. According to [1], b should be selected to guarantee that $\pi(\theta|b)$ is a decreasing function of θ. If we take the conjugate prior (4), hyperparameter b should be in the ranges $b > 0$, due to $\frac{d\pi(\theta|b)}{d\theta} < 0$. Prior distribution with thinner tail would make worse robustness of Bayesian distribution. Accordingly, b should not too big. It is better to choose $0 < b < c$ ($c > 0$, and c is a constant).

Suppose that the prior distributions of b are uniform distribution in $(0,c)$. Therefore, the prior distribution of b is given by

$$\pi(b) = \frac{1}{c}, \qquad 0 < b < c. \tag{13}$$

The definition of E-Bayesian estimation of Reliability is described using the definition of [1] as follows.

Definition 3.1. The E-Bayesian estimation of the reliability function, R, is the expectation of the Bayesian estimation of R for the hyperparameter b and is defined as

$$\hat{R}^{EBk}(\mathbf{x}) = \int_D \hat{R}^{Bk}(\mathbf{x})\pi(b)db = E(\hat{R}^{Bk}(\mathbf{x})), \tag{14}$$

where $\pi(b)$ is the prior density function of hyperparameter b and D is the domain of b.

In the following theorem, we obtain the E-Bayesian estimators of R under the loss function (3) and the prior distribution (13).

Theorem 3.2. *Let $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ be the sample observations from the exponential distribution. Then, the E-Bayesian estimator of R corresponding to the prior given in (13) under the loss function (3) are equal to*

$$\hat{R}^{EBk}(\boldsymbol{x}) = \frac{1}{c}\sum_{i=0}^{n+1}\binom{n+1}{i}\frac{(-t)^i}{1-i}[(S+c+t-tk)^{1-i} - (S+t-tk)^{1-i}]. \tag{15}$$

Proof. For prior $\pi(b)$, the E-Bayesian estimator of R under the loss function (3) is given by

$$\hat{R}^{EBk}(\mathbf{x}) \;=\; \int_0^c \frac{1}{c}[\frac{S+b-tk}{S+b+t-tk}]^{n+1}db$$

$$= \frac{1}{c}\int_{S+t-tk}^{S+c+t-tk}[1-\frac{t}{y}]^{n+1}dy$$
$$= \frac{1}{c}\int_{S+t-tk}^{S+c+t-tk}\sum_{i=0}^{n+1}\binom{n+1}{i}(\frac{-t}{y})^{i}dy$$
$$= \frac{1}{c}\sum_{i=0}^{n+1}\binom{n+1}{i}(-t)^{i}\int_{S+t-tk}^{S+c+t-tk}y^{-i}dy$$
$$= \frac{1}{c}\sum_{i=0}^{n+1}\binom{n+1}{i}\frac{(-t)^{i}}{1-i}[(S+c+t-tk)^{1-i}-(S+t-tk)^{1-i}],$$

where $y=S+b+t-tk$. which ends the proof. □

4 E-Posterior risk of Bayesian estimators

[5] proposed the definition of E-posterior risk to measure the estimated error of E-Bayesian estimators.

Definition 4.1. The E-Posterior risk of $\hat{R}^{Bk}$ is the expectation of the posterior risk of the Bayesian estimator for the hyperparameter b which is defined as

$$EP(\hat{R}^{EBk}) = \int_{D}\rho(\pi,\hat{R}^{Bk})\pi(b)db = E(\rho(\pi,\hat{R}^{Bk})).$$

In the following theorem, we obtain the E-Posterior risk under the loss function (3) and prior distributions given in (13).

Theorem 4.2. *Let* $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ *be the sample observations from the exponential distribution. Therefore, we have:*
(i) The posterior risk of $\hat{R}^{Bk}(\boldsymbol{x})$ *under the loss function (3) is given by*

$$\rho(\pi,\hat{R}^{Bk}) = [\frac{S+b}{S+b+2t-kt}]^{n+1}-\frac{[(S+b)(S+b-tk)]^{n+1}}{[S+b+t-kt]^{2(n+1)}}. \qquad (16)$$

(ii)The E-Posterior risk of $\hat{R}^{Bk}$ *under the loss function (3) is given by*

$$EP(\hat{R}^{EBk}) = \frac{1}{c}\int_{0}^{c}[\frac{S+b}{S+b+2t-kt}]^{n+1}-\frac{[(S+b)(S+b-tk)]^{n+1}}{[S+b+t-kt]^{2(n+1)}}db. \qquad (17)$$

Proof. (i) From (8), the posterior risk of $\hat{R}^{Bk}(\mathbf{x})$ under the loss function (3) is given by

$$\rho(\pi,\hat{R}^{Bk}) = E[R^{2-k}|\mathbf{x}] - 2\hat{R}^{Bk}E[R^{1-k}|\mathbf{x}] + (\hat{R}^{Bk})^2E[R^{-k}|\mathbf{x}]$$

Using the relations (3) and (11) and also

$$E[R^{2-k}|\mathbf{x}] \;=\; [\frac{S+b}{S+b+t(2-k)}]^{n+1}.$$

we have

$$\begin{aligned}\rho(\pi,\hat{R}^{Bk}) &= [\frac{S+b}{S+b+t(2-k)}]^{n+1} - 2[\frac{(S+b-tk)(S+b)}{(S+b+t-tk)(S+b+t(1-k))}]^{n+1}\\ &+ [\frac{S+b-tk}{S+b+t-tk}]^{2(n+1)}[\frac{S+b}{S+b-tk}]^{n+1}\\ &= [\frac{S+b}{S+b+2t-kt}]^{n+1} - \frac{[(S+b)(S+b-tk)]^{n+1}}{[S+b+t-kt]^{2(n+1)}}.\end{aligned}$$

(ii) For prior distribution $\pi(b)$, the E-posterior risk of $\hat{R}^{Bk}$ under the loss function (3) is given by

$$\begin{aligned}EP(\hat{R}^{EBk}) &= \int_0^c \rho(\pi,\hat{R}^{Bk})\pi(b)db\\ &= \frac{1}{c}\int_0^c \{[\frac{S+b}{S+b+2t-kt}]^{n+1} - \frac{[(S+b)(S+b-tk)]^{n+1}}{[S+b+t-kt]^{2(n+1)}}\}db.\end{aligned}$$

□

5 Simulation study

In this section, we perform a Monte Carlo simulation for comparison of the E-Bayesian estimators of reliability function R and its E-Posterior risk. For this purpose, we generate sequences n of independent random samples from the exponential distribution with $\theta = 2$. If we consider $t = 1$, then the true value of reliability function is $R = e^{-\theta t} = 0.1353$.

The performance of estimates of reliability function has been compared for repeated $N = 10000$ times simulation runs in terms of $\hat{R}^{EBk}$, $k = 0,1,2$

and $EP^{(k)} = EP(\hat{R}^{EBk}),\ k = 0,1,2,$ for selected values of $c = 0.5,1,3,5$ and $n = 20,40,70$. The results are summarized in Table 1. From Table 1 we conclude that:

1. The following relationship can be observed between the values of E-Bayesian estimates:

$$\hat{R}^{EB2} < \hat{R}^{EB1} < \hat{R}^{EB0}.$$

2. The values $EP(\hat{R}^{EBk}),\ k = 0,1,2$ have the following relationship

$$EP(\hat{R}^{EB0}) < EP(\hat{R}^{EB1}) < EP(\hat{R}^{EB2}),$$

which suggests that, if the E-Posterior risk is used as an evaluation measure, then $\hat{R}^{EB0}$ is superior to $\hat{R}^{EB1}$ and $\hat{R}^{EB1}$ is superior to $\hat{R}^{EB2}$.

3. The values of $EP(\hat{R}^{EBk}),\ k = 0,1,2$ decreases as the sample size increase.

Table 1: Results of E-Bayesian estimates and its E-Posterior risk.

n	c	$\hat{R}^{EB0}$	$\hat{R}^{EB1}$	$\hat{R}^{EB2}$	$EP^{(0)}$	$EP^{(1)}$	$EP^{(2)}$
20	0.5	0.1447301	0.120392	0.096938	0.003543	0.024337	0.246662
	1	0.150884	0.126442	0.102725	0.003681	0.024442	0.234651
	3	0.175418	0.150941	0.126683	0.004187	0.024477	0.195971
	5	0.199378	0.175287	0.151084	0.004607	0.024091	0.168000
40	0.5	0.142985	0.130169	0.117456	0.001835	0.012816	0.110700
	1	0.146197	0.133369	0.120626	0.001874	0.012828	0.107904
	3	0.159030	0.146203	0.133397	0.002023	0.012827	0.098000
	5	0.171766	0.159006	0.146217	0.002159	0.012759	0.089764
70	0.5	0.138614	0.131096	0.123597	0.001042	0.007517	0.060600
	1	0.140495	0.132974	0.125470	0.001056	0.007520	0.059730
	3	0.148016	0.140496	0.132980	0.001109	0.007519	0.056487
	5	0.155515	0.148010	0.140498	0.001161	0.007504	0.053579

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A Note on the Generalized Mixed Shock Models

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Abstract: Suppose that a system is affected by a sequence of shocks that occur randomly over time, and δ_1, δ_2, η_1 and η_2 are critical levels such that $0 < \delta_1 < \delta_2$ and $0 < \eta_1 < \eta_2$. In this paper, a new mixed δ-shock model is introduced for which the system fails with a probability, say θ_1, when the time between two consecutive shocks lying in $[\delta_1, \delta_2]$, and the system fails with a probability, say θ_2, when the magnitude of a shock lying in $[\eta_1, \eta_2]$. The system fails with probability 1, as soon as the interarrival time between two successive shocks is less than δ_1 or a shock with magnitude greater than η_2 occurs. Under this model, the corresponding conditional distributions which is needed to determine the survival function is derived.

Keywords: δ-shock models, Shock models, Survival function.

1 Introduction

Different types of the shock models have been used in the reliability theory. In the mixed shock models, two or more types of classic shocks affect the performance of the system (for example, mixed of extreme shock model and δ-shock model). The mixed shock models are studied by [12], [1], [3],

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[11], [9], [6], [7], [8]. [4] studied the behavior of a new system, in which the system fails, when a shock with magnitude bigger than c_2 is entered on the system and he let the system working by less performance if the magnitude of the shock is in (c_1,c_2), for $c_1 < c_2$. [11] studied reliability modeling for systems subject to dependent competing risks considering the impact from a new generalized mixed shock model. The work of [9] is extended for the multi-state system by [6], where three main reasons for the failure of system are considered: the number of times between two consecutive shocks which are in $[\delta_1,\delta_2]$ is k, a time between two consecutive shock is less than δ_1 and finally, a shock with magnitude greater than γ is entered to the system. Recently, ([7, 8]) studied a system, in which the system fails when the number of times between two consecutive shocks which are less than δ is k, or the magnitude of the shock is greater than γ. According to the model introduced by [5], for two fixed critical values d_1 and d_2 such that d_1¡d_2, the system under concern fails upon the occurrence of k consecutive shocks of size at least d_1 or a single large shock of size at least d_2. The run and extreme shock models are mixed in this model.

In this paper, we let the entered shocks to the system have a random magnitude and the time between two consecutive shocks are also random. The corresponding conditional distributions which is needed to determine the survival function is then provided.

2 The behavior of the systems lifetime

Let the entered shocks to the system have a random magnitude. We also assume that the time between two consecutive shocks are random. If the time between two successive shocks lying in $[\delta_1,\delta_2]$, for 0¡δ_1¡δ_2, we let the system fails with probability θ_1 $(0 \leq \theta_1 \leq 1)$ and if the magnitude of a shock lying in $[\eta_1,\eta_2]$, for 0¡η_1¡η_2, we let the system fails with probability θ_2 $(0 \leq \theta_2 \leq 1)$. The system fails with probability 1, as soon as the interarrival time between two successive shocks is less than δ_1 or the magnitude of a shock is larger than η_2. Also, we let, the system does not fail when the interarrival time between two successive shocks is greater than δ_2 or the magnitude of shock is less than η_1. Corresponding to each interarrival time in $[\delta_1,\delta_2]$, we introduce a random variable Y obeying Bernoulli distribution with probability of success $\theta_1$and corresponding to each magnitude of shock in $[\eta_1,\eta_2]$, we introduce a random variable W obeying Bernoulli distribution with probability of success θ_2.

In this paper, we let, if the interarrival time between two successive shocks is less than δ_1, this interarrival time is critical, and if the interarrival time between two successive shocks is lying in $[\delta_1,\delta_2]$, this interarrival time is probably critical, and if the interarrival time between two successive shocks is larger than δ_2, this interarrival time is non-critical. Also if the magnitude of a shock is larger than η_2, this shock is critical, and if the magnitude of a shock is lying in $[\eta_1,\eta_2]$, this shock is probably critical, and if the magnitude of a shock is less than η_1, this shock is non-critical. According to these definitions, we can write N_1,N_2,N_3,N_4, by the following:

N_1: The number of non-critical interarrival times between two successive shocks and non-critical shocks that the system encounters before failure.

N_2: The number of probably critical interarrival times between two successive shocks and non-critical shocks that the system encounters before failure.

N_3: The number of non-critical interarrival times between two successive shocks and probably critical shocks that the system encounters before failure.

N_4: The number of probably critical interarrival times between two successive shocks and probably critical shocks that the system encounters before failure.

Then, $T_{\theta_1,\theta_2} = \sum_{i=1}^{N} U_i$ is the lifetime of the system, where $N = \sum_{j=1}^{4} N_j + 1$

is the number of impact times between two shocks until failure of system. Set

$$
\begin{aligned}
p_1 &= \bar{F}(\delta_2) - \bar{H}(\delta_2, \eta_1),\\
p_2 &= (H(\delta_2, \eta_1) - H(\delta_1, \eta_1))(1-\theta_1),\\
p_3 &= (\bar{H}(\delta_2, \eta_1) - \bar{H}(\delta_2, \eta_2))(1-\theta_2),\\
p_4 &= ((\bar{H}(\delta_1, \eta_1) - \bar{H}(\delta_2, \eta_1)) - (\bar{H}(\delta_1, \eta_2) - \bar{H}(\delta_2, \eta_2)))(1-\theta_1)(1-\theta_2),\\
p_5 &= 1-(p_1+p_2+p_3+p_4),
\end{aligned}
$$

The joint probability (mass) function of (N_1, N_2, N_3, N_4) and the probability (mass) function of N are given in the following lemma.

Lemma 2.1. *The joint probability (mass) function of* (N_1, N_2, N_3, N_4) *is given by*

$$
\begin{pmatrix} P(N_1 = n_1, N_2 = n_2, N_3 = n_3, N_4 = n_4) = \sum_{i=1}^{4} n_i \\ n_1, n_2, n_3, n_4 p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} p_5, \end{pmatrix}
$$

and the probability (mass) function of N is given by

$$
P(N = n) = (1-p_5)^{n-1} p_5.
$$

3 The conditional distributions

In this section, the corresponding conditional distributions which is needed to compute the survival function is derived. Consider the conditional distributions $F^{(1)}(u) := P(U \le u | U > \delta_2, Z < \eta_1)$, $F^{(2)}(u) := P(U \le u | \delta_1 < U < \delta_2, Z < \eta_1)$, $F^{(3)}(u) := P(U \le u | U > \delta_2, \eta_1 < Z < \eta_2)$, and $F^{(4)}(u) := P(U \le u | \delta_1 < U < \delta_2, \eta_1 < Z < \eta_2)$.

Theorem 3.1. *We have*

$$
\begin{aligned}
F^{(1)}(u) &= \frac{H(u, \eta_1) - H(min(u, \delta_2), \eta_1)}{\bar{F}(\delta_2) - \bar{H}(\delta_2, \eta_1)}\\
F^{(2)}(u) &= \frac{H(min(\delta_2, u), \eta_1) - H(\delta_1, \eta_1}{H(\delta_2, \eta_1) - H(\delta_1, \eta_1)}
\end{aligned}
$$

$$
\begin{aligned}
F^{(3)}(u) &= \frac{\bar{H}(\delta_2,\eta_1)-\bar{H}(u,\eta_1)-(\bar{H}(\delta_2,\eta_2)-\bar{H}(u,\eta_2))}{\bar{H}(\delta_2,\eta_1)-\bar{H}(\delta_2,\eta_2)} \\
F^{(4)}(u) &= \frac{\bar{H}(\delta_1,\eta_1)-\bar{H}(min(\delta_2,u),\eta_1)-(\bar{H}(\delta_1,\eta_2)-\bar{H}(min(\delta_2,u),\eta_2))}{\bar{H}(\delta_1,\eta_1)-\bar{H}(\delta_2,\eta_1)-(\bar{H}(\delta_1,\eta_2)-\bar{H}(\delta_2,\eta_2))}
\end{aligned}
$$

Proof. We prove this theorem only for $F^{(1)}(u)$. The proofs of other cases are similar. According to definition of $F^{(1)}(u)$, we have

$$
\begin{aligned}
F^{(1)}(u) &= P(U \leq u | U > \delta_2, Z < \eta_1) \\
&= \frac{P(U \leq u, U > \delta_2, Z < \eta_1)}{P(U > \delta_2, Z < \eta_1)} \\
&= \frac{P(\delta_2 < U \leq u, Z < \eta_1)}{P(U > \delta_2, Z < \eta_1)} \\
&= \frac{H(u,\eta_1)-H(min(u,\delta_2),\eta_1)}{\bar{F}(\delta_2)-\bar{H}(\delta_2,\eta_1)}.
\end{aligned}
$$

□

Example 3.2. Suppose that 10 minutes is enough time to do customer service in a bank teller, but in 5 percent of cases, for various reasons such as high customer workload, 10 to 15 minutes is needed, so if the time interval between customer arrivals is less than 10 minutes, customer service will have problems, but if this distance is more than 15 minutes, there will be no problem. Also, if the time interval between consecutive inputs is between 10 and 15 minutes, in 5 percent of cases, service and performance will be disrupted.

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On the Regression Modeling of Survival Data Using Pseudo-observations

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Abstract: Pseudo-observations have been introduced as an approach to perform regression modeling of a mean value parameter with right censored survival data. Once the pseudo-observations have been computed, the models can be fit by using generalized estimating equation. The pseudo-observations with the focus on survival function are introduced. Analyzing regression models based on pseudo-observations is then discussed. A simulation study is conducted to illustrate the use of pseudo-observations in regression analysis on the survival function.

Keywords: GEE, Kaplan-Meier curves, Pseudo-observations, Regression analysis, Restricted mean survival.

1 Introduction

In many applied settings such as medicine, biology, epidemiology, economics, and demography, the outcome is time to some event of interest, which is often incompletely observed due to censoring. To evaluate the effects of covariates on such an outcome, Cox proportional hazards model ([4]) is frequently used in most applications. This regression model is specified via the hazard function. Inference on the Cox model is based on a

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partial likelihood approach, in which the problem with censored data is handled by putting most emphasis on the observed most emphasis on the observed event times. In fact, presence of censoring put restriction on analysis of survival data. Without censored survival data, the survival time (outcome) would be observed for all individuals and standard methods could be used to analyze such data. Pseudo-observations can be considered as a tool for analyzing survival data. This approach were first suggested by [3] for performing generalized linear regression analysis of survival data. The pseudo-observation technique allows direct regression modeling of the survival function ([8]), the restricted mean survival time ([2]), and the cumulative incidence function for competing risks data ([7]). The approach uses the pseudo-observations based on jackknife estimates that show the contribution of an individual to the non-parametric estimator of the parameter of interest. The pseudo-observations calculate for each individual in the sample as the difference between the complete sample and the leave-one-out estimators of relevant survival quantities. These pseudo-observations are then used in a generalized estimating equation (GEE) to estimate the effects of covariates on the outcome of interest.

The structure of the paper is organized as follows. In Section 2, we introduce pseudo-observations specially for survival probabilities. Section 3 discusses analyzing regression models based on pseudo-observations with the focus on survival function as a parameter of interest. Section 4 reports some results from a simulation study conducted to illustrate the use of pseudo-observations in regression analysis on the survival function. Section 5 contains some discussion and concluding remarks.

2 The pseudo-observation method

Let $T_1, \ldots, T_n$ be independent and identically distributed lifetime random variables. Under right-censoring, the observed dataset consists of n independent identically distributed pairs $(\tilde{T}_i, \delta_i)$, where $\tilde{T}_i$ is the observed time and δ_i is the censoring indicator for the ith individual. Let θ be a parameter of the form $\theta = E[f(T_i)]$ for some function $f(.)$ which may be multivariate. Suppose an unbiased (or approximately unbiased) estimator $\hat{\theta}$ for the parameter θ is available, that is $E[\hat{\theta}] = \theta$. In addition, Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent and identically distributed covariates where each $\mathbf{X}_i = [X_{i1}, \ldots, X_{ip}]^\top$. The conditional expectation of $f(T_i)$ given $\mathbf{X}_i$ is defined by

$$\theta_i = E[f(T_i) \mid \mathbf{X}_i].$$

Jackknife theory can then be used to define the pseudo-observation for $f(T)$ for individual i, $i = 1, \ldots, n$, as

$$\hat{\theta}_i = n \,.\, \hat{\theta} - (n-1) \,.\, \widehat{\theta^{-i}},$$

where $\widehat{\theta^{-i}}$ is the "leave-one-out" estimator for θ based on T_j, $j \neq i$. In other words, $\widehat{\theta^{-i}}$ is the estimator obtained from the sample size $n-1$ by removing the ith individual from the study.

The pseudo-observations $\hat{\theta}_i$ will always be used for all n (censored and uncensored) individuals in the study. It should be noted that if the data were complete (i.e. all T_i are observed), then θ could be estimated by $\sum_i f(T_i)/n$ that in this case $\hat{\theta}_i$ is simply $f(T_i)$. Therefore, this approach should be used in a case that only a censored sample of the T_i is available.

To illustrate this approach, we consider the survival probability $S(t) = P(T > t) = E[I(T > t)]$ as the mean value parameter of interest. At a fixed time point t_0, the function of interest $f(.)$ is given by

$$f(T) = f_t(T) = I(T > t),$$

and the parameter θ is the survival function evaluated at t_0, $S(t_0)$. Since the Kaplan-Meier estimator $\hat{S}(t)$([6]) is approximately unbiased estimator of $\theta = S(t)$ ([1]), the ith pseudo-observation is then given by

$$\hat{S}_i(t_0) = n \,.\, \hat{S}(t_0) - (n-1) \,.\, \widehat{S^{-i}}(t_0),$$

where $\widehat{S^{-i}}(t_0)$ is the Kaplan-Meier estimator $S(t_0)$ based on $n-1$ observations $j \neq i$. At a grid of fixed time points $t_1 < \ldots < t_m$, as a multivariate version, the function $f(.)$ is

$$f(T) = [f_{t_1}(T), \ldots, f_{t_m}(T)] = [I(T > t_1), \ldots, I(T > t_m)],$$

with parameters

$$\theta = [\theta_1, \ldots, \theta_m] = [S(t_1), \ldots, S(t_m)].$$

When θ is a multivariate parameter of dimension m, then for each individual t, m pseudo-observations can be defined as the following

$$\hat{\theta}_{ik} = \hat{S}_i(t_k) = n\,.\,\hat{S}(t_k) - (n-1)\,.\,\widehat{S^{-i}}(t_k), \quad k = 1, \ldots, m.$$

3 Regression models based on pseudo-observations

In survival analysis, it often is interesting to study the association between the survival experience of an individual and some covariates using regression models which are often based on the hazard function. However, in some situations more general regression models might be desirable where in the presence of censored data, standard methods for modeling do not exist. Pseudo-observations provide a common approach to different kinds of models using generalized linear regression analysis in survival data.

A regression model to specify the relationship between θ_i and $\mathbf{X}_i$ can be provided by a generalized linear model as

$$g(\theta_i) = g(E[f(T_i) \mid \mathbf{X}_i]) = \beta^\top \mathbf{X}_i,$$

where $g(.)$ is some link function. Here a column $X_{i0} = 1$ is added to $\mathbf{X}_i$ which contributes to an intercept β_0. [3] suggested to replace the function $f(.)$ by a pseudo-observation, and then estimate the unknown parameters by using the GEE based on the pseudo-observations. It should be noted that for each individual i, if the parameter θ_i is m-dimensional (m time points), $\theta_i = [\theta_{i1}, \cdots, \theta_{im}]^\top$, then for each $\theta_{ik}, k = 1, \cdots, m$, a model can be specified as

$$g(\theta_{ik}) = g(E[f_{t_k}(T_i) \mid \mathbf{X}_i]) = \beta^\top \mathbf{X}_{ik},$$

where the vector $\mathbf{X}_{ik}$ includes indicators of the time points, $I(t_l = t_k), l = 1, \cdots, m$, to allow for different intercepts at each time. Therefore, here, the parameter β is $m + p$-dimensional which can be estimated from the following generalized estimating equations

$$U(\beta) = \sum_{i=1}^{n} U_i(\beta) = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \beta} g^{-1}(\beta^\top \mathbf{X}_{ik})\right]^\top \mathbf{V}_i^{-1} \left[\hat{\theta}_i - g^{-1}(\beta^\top \mathbf{X}_{ik})\right] = \mathbf{0}, \quad (1)$$

where $\mathbf{V}_i$ is a $k \times k$ working covariance matrix for $\hat{\theta}_i$.

Let $\hat{\beta}$ denote the solution to (1). The covariance matrix for $\hat{\beta}$ can be estimated by the standard sandwich estimator as

$$\widehat{var}(\hat{\beta}) = I(\hat{\beta})^{-1} \widehat{var}(U(\hat{\beta})) I(\hat{\beta})^{-1},$$

where

$$\begin{aligned} I(\beta) &= \sum_{i=1}^{n} \left[\frac{\partial}{\partial \beta} g^{-1}(\beta^\top \mathbf{X}_{ik})\right]^\top \mathbf{V}_i^{-1} \left[\frac{\partial}{\partial \beta} g^{-1}(\beta^\top \mathbf{X}_{ik})\right], \\ \widehat{var}(U(\beta)) &= \sum_{i=1}^{n} U_i(\beta)^\top U_i(\beta). \end{aligned}$$

A non-parametric bootstrap technique can also be used to find an alternative jackknife estimator for the covariance of $\hat{\beta}$ ([10]). Based on GEE results of [9], [5] showed the estimated regression parameters $\hat{\beta}$ are asymptotically normal and consistent estimators of β.

Note that the number and position of time points, m, is a choice which must be made prior to the analysis. However, a single time point, $m = 1$, is enough to obtain estimates of the regression parameters, more time points may be improve efficiency for capturing the trend in the event distribution. According to the results obtained by [2] and [7], increasing m does not have a significant impact on the precision of the estimated regression parameters. They also suggested that five to ten time points equally spaced on the event time scale works quite well in most cases. Now suppose we are interested to perform regression analysis on the survival function. In this case, as discussed in the previous section, at a single time point t_0, we have $f(T) = I(T > t_0)$ and thereby $\theta = S(t_0)$. Then choosing the link function as the cloglog-function $g(x) = c\log\log(x) = \log(-\log(x))$, the generalized linear model becomes

$$\log\Big(-\log(S(t_0 \mid \mathbf{X}_i))\Big) = \beta_0 + \beta^\top \mathbf{X}_i, \quad i = 1, \cdots, n \tag{2}$$

with $\beta_0 = \log(H_0(t_0))$ where $H_0(t) = \int_0^t h_0(u)du$ is the cumulative baseline hazard function. The model (2) corresponds to the Cox proportional hazards model with survival function

$$S(t \mid \mathbf{X}_i) = \exp\Big\{-H_0(t)\exp\Big(\beta^\top \mathbf{X}_i\Big)\Big\}.$$

The model (2) can be extended to a joint proportional hazards model for a grid of time points $t_1, \cdots, t_m$ as

$$\log\Big(-\log(S(t_k \mid \mathbf{X}_i))\Big) = \beta_{0k} + \beta^\top \mathbf{X}_i, \quad k = 1, \cdots, m \tag{3}$$

where $\beta_{0k} = \log(H_0(t_k))$ indicates that the intercept may depend on the time point t_k.

4 Simulation study

A simulation study was conducted to illustrate the use of pseudo-observations in regression analysis of model (3). Survival data were generated from the Cox model

$$h(t \mid x_1, x_2) = h_0(t)\exp(\beta_1 x_1 + \beta_2 x_2), \tag{4}$$

with $h_0(t) = t$, $(\beta_1, \beta_2)^\top = (3, -1)^\top$, and x_1 is a Bernoulli random variable with success probability 0.5 which is independent of x_2 that is a standard normal distributed variable. Independent censoring times were generated from the uniform distribution on $(0, c)$ where the constant c was selected to result in, on average, about 10% or 25% of observations censored. For each individual the pseudo-observations were calculated at equally spaced time points within the range of the observed times. Each configuration was based on 1000 replications with sample sizes $n = 100$ and 300, and for each replication the model (3) was fitted. For comparison, the Cox model (4) was also fitted to the data.

Table 1 summarizes the simulation results for the estimates of β_1 and β_2 based on $m = 5$ time points. The table shows the average of the 1000 estimated regression parameters ($Est.$) and their corresponding standard errors obtained from sandwich estimator $\widehat{var}(\hat{\beta})$ (SEE_{san}) or approximate jackknife variance estimator (SEE_{aj}). The average of the standard errors of the regression parameters (SE) is also shown in table 1. The results indicate that the GEE model based on the pseudo-observations seems to perform quite well. Some bias is seen for the estimated regression parameters, but it diminishes when the sample size is increased. The estimated standard errors are close to the empirical standard errors. Comparing with the Cox model, both the estimated standard errors and the empirical standard errors seem to be higher for the model based on the pseudo-observations.

Discussion and conclusions

Jackknife pseudo-observations have been studied as a tool for analyzing right-censored survival data. This approach is based on a set of pseudo-observations computed for each individual in the study. The GEE approach based on pseudo-observations can be used to analyze the effect of potential regression covariates for functions on the event times. The advantage of this approach is that it allows one to model the event times by generalized linear models without specifying a full parametric model. A sim-

Table 1: Summary of simulation results.

		Pseudo-observations				Cox model		
	$C\%$	$Est.$	SE	SEE_{san}	SEE_{aj}	$Est.$	SE	SEE
$n = 100$								
β_1	10	3.253	0.624	0.538	0.543	3.035	0.315	0.358
	25	3.279	0.839	0.645	0.864	2.986	0.376	0.374
β_2	10	-1.071	0.237	0.223	0.224	-1.012	0.139	0.144
	25	-1.064	0.308	0.260	0.261	-0.986	0.159	0.154
$n = 300$								
β_1	10	3.066	0.285	0.286	0.286	3.008	0.182	0.199
	25	3.049	0.285	0.325	0.324	3.023	0.205	0.211
β_2	10	-1.015	0.122	0.121	0.121	-0.991	0.075	0.079
	25	-1.022	0.135	0.139	0.139	-1.013	0.091	0.086

$C\%$: the censoring percentage; $Est.$: the estimate of the parameter; SE: sample standard error of the estimates; SEE: the mean of the standard error of the estimates.

ulation study was conducted to compare the traditional Cox proportional hazards model with regression analysis based on pseudo observations. The results of simulation study showed that the estimates for the Cox model based on pseudo-observations are more dispersed than those based on the standard partial likelihood methods. In fact, this approach provides a way of inference for cases that standard methods may not be available.

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A Two-stage Approach for Joint Modeling of Longitudinal Measurements and Time-to-event Data in the Presence of Multiple Failure Types

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Abstract: In many practical studies of medical sciences it is common to use the joint modeling of longitudinal measurements and time-to-event data. Most of the time, particularly in clinical studies and health inquiry, there are more than one event and they compete for failing an individual. In this situation assessing the competing risk failure time is important. In most cases, implementation of joint modeling involves complex calculations. Therefore, in this paper, we propose a two-stage method for joint modeling of longitudinal measurements and competing risks (JMLC) data based on the full likelihood approach via the conditional EM (CEM) algorithm. In the first stage, a linear mixed effect model is used to estimate parameters of the longitudinal sub-model. In the second stage, we consider a cause-specific sub-model to construct competing risks data and to describe an approximation for the joint log-likelihood that uses the estimated parameters of the first stage. Finally, we perform this method on the "standard and new anti-epileptic drugs" trial to check the effect of drug

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1 Introduction

assaying on the treatment effects of lamotrigine and carbamazepine through treatment failure. In analyzing follow-up data, we are faced with a lot of issues that deal with joint modeling of longitudinal and time-to-event data. Hence, a lot of researches have been grown in this regard over the recent years . The standard joint model includes two sub-models; a longitudinal sub-model and a time-to-event sub-model. These two sub-models are related through an association structure that measures the relationship between the favorite outcomes in the study. Commonly, a linear-mixed effects sub-model for the longitudinal process is used for analyzing longitudinal outcome and a Cox proportional hazard sub-model is considered for the time-to-event process.

The relevant literature documents studies used the likelihood approach to estimate the parameters in these models based on the shared random effects model. In order to implement the full likelihood approach, both the classical and the Bayesian paradigms are used in the literature. Although the use of the Bayesian paradigm partially simplifies computations, the existence of multi-integrals in the joint log-likelihood function and the survival function still need complex calculations. Therefore, a two-stage method is suggested which is performed in two stages; in the first stage, the longitudinal data are fitted and in the second stage the fitted values of the longitudinal process are used as covariates in the joint model to estimate the survival parameters. Therefore, the two-stage methods can solve the problem of complex calculation in the full likelihood approaches. [3] propose a modified two-stage approach to reduce the biases in this approach. Most of the studies employing joint modeling framework focus on data with a single event time and only one failure cause. However, in medical research and some situations of interest, there are more than one possible cause of event or the censoring is informative. In such cases, the subjects of the study are at risk with more than one mutually exclusive event such as death from different causes, so that competing risks data arise naturally. Studies on joint modeling of longitudinal measurements and competing risks time-to-event data have grown in the past decade.

In the following, we will refer to two papers in this regard. [2] proposed a joint model for longitudinal measurements and competing risks survival data and developed a Bayesian MCMC procedure for parameter estimation and inference. [4] provided a comparison of joint models for longitudinal and competing risks data and summarized four published models. In fact They comprehensively reviewed the literature for implementation of joint models involving more than one event time per subject.

In this paper, we propose a two-stage approach for joint modeling of longitudinal measurements and competing risks data. Because of two-stage structure, it facilities computational problems of shared random effects in joint models and makes it possible to use the standard packages of mixed-effects model and survival models in R software.

2 Main results

2.1 Modeling framework

Consider a longitudinal study with n individuals. These recorded longitudinal responses are measured at specific times s_{ij}, $i = 1,2,...,n$, $j = 1,2,...,n_i$, therefore $y_{ij} = y_i(s_{ij})$ denotes longitudinal measurement for the ith subject at time s_{ij}. Also, the observed failure time for the ith subject is the minimum of true survival time and the censoring time and it is denoted by $T_i = min(T^*_{i1}, T^*_{i2}, ..., T^*_{iK}, C_i)$; where T^*_{ik} is the true survival time of subject

i for each event type $k = 1, 2, ..., K$ and C_i is the randomly censoring time for the ith subject. Also, δ_i is defined as the event indicator, which takes value $\{0, 1, 2, ..., K\}$, with 0 corresponding to censoring and $1, 2, ..., K$ to the competing events.

In order to construct a two-stage approach for joint modeling of longitudinal measurements and competing risks data, we consider two sub-models as described separately in the following section; the association of these models is also discussed.

2.1.1 Longitudinal sub-model

We assume a linear mixed effect sub-model to analyze longitudinal data. The longitudinal sub-model can be written as:

$$Y_i(s) = m_i(s) + \varepsilon_i(s), \; i = 1, 2, ..., n, \tag{1}$$

where, $m_i(s)$ is the mean response term for the ith subject and it is modeled as:

$$m_i(s) = X_i^{'}(s)\beta + Z_i^{'}(s)b_i, \tag{2}$$

and $\varepsilon_i(s) \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$ is the error term for the ith subject at time s. $X_i(s)$ is a $n_i \times p$ design matrix of the p observed explanatory variables and $Z_i(s)$ is a $n_i \times q$ design matrix of the q random effects for the ith subject. β is a p-dimensional vector of fixed effects and b_i is a q-dimensional vector of random effects such that $b_i \sim N_q(0, D)$, where D is a positive define matrix.

2.1.2 Competing risks sub-model

We consider the distribution of the competing risks failure time (T_i, δ_i) takes the form of the following cause-specific hazard frailty model:

$$\begin{aligned} h_{ik}(t|\mathscr{M}_i(t), w_i) &= lim_{dt \to 0} P\{t \leq T_i < t + dt | T_i \geq t, \mathscr{M}_i(t), \delta_i = k, w_i\}/dt \\ &= h_{0k}(t)\, exp\{\gamma_k^{'} w_i + \alpha_k m_i(t)\}. \end{aligned} \tag{3}$$

Here, $h_{0k}(t)$ is the baseline hazard function and w_i is the vector of covariates. $\mathcal{M}_i(t) = \{m_i(s), 0 \leq s < t\}$ shows the history of the true unobserved longitudinal process up to time t. Also, γ_k is a p_k-dimensional vector of cause-specific regression coefficients for w_i and $\alpha = (\alpha_1, ..., \alpha_K)^{'}$ is a K-dimensional vector such that α_k, $k = 1, ..., K$ is a coefficient for the shared time-variating covariate $m_i(t)$ that is defined in longitudinal sub-model and these two sub-models are related through it.

2.1.3 Joint model structure

Note that, we can construct a joint model of longitudinal and time-to-event data for each event type because of the choice of cause-specified hazard for competing risks data. Therefore, to construct a joint model, we need a full joint distribution of both process. There are different factorization for this joint distribution that leads to different approaches for modeling.

In many studies concerning joint modeling of longitudinal measurements and time-to-event data, it is common to use a shared parameter model. Usually, standard methods use a linear-mixed effects sub-model for the longitudinal process and a Cox proportional hazard sub-model for the time-to-event process.

Here, we consider a shared parameter model to join the longitudinal process to competing risks process in which the longitudinal process and time-to-event process for the kth failure type are linked by the parameter α_k.

According to the two defined sub-models (1) and (2), the hazard rate in competing risks model depends on the true unobserved longitudinal measurements at time t. Also, we suppose that the longitudinal sub-model and the survival process in the competing risks sub-model related to the vector of random effects b_i, therefore, under these assumptions, we can write the likelihood function of the joint model. Let $Y = (Y_1^{'}, ..., Y_n^{'})^{'}$, $Y_i = (Y_{i1}, ..., Y_{in_i})^{'}$, $b = (b_1^{'}, ..., b_n^{'})^{'}$, $t = (t_1, ..., t_n)^{'}$, $\delta = (\delta_1, ..., \delta_n)^{'}$ and

θ be the vector of all parameters in two sub-models, then given the random effects b_i, the elements of y_i and (t_i, δ_i) are independent and we have:

$$P(t_i, \delta_i, y_i | b_i; \theta) = \prod_{j=1}^{n_i} P(y_{ij} | b_i; \theta) P(t_i, \delta_i | b_i; \theta). \quad (4)$$

Therefore the log-likelihood function of joint models based on the observed data can be written as follows:

$$\begin{aligned} l(\theta | y, t, \delta) &= \sum_i \log P(t_i, \delta_i, y_i; \theta) \\ &= \sum_i \log \int_{b_i} P(t_i, \delta_i, y_i, b_i; \theta) db_i \\ &= \sum_i \log \int_{b_i} P(t_i, \delta_i | b_i; \theta_t, \beta) P(y_i | b_i; \theta_y) P(b_i; \theta_b) db_i, \; \delta_i = 1, 2, ..., K, \end{aligned}$$

where $\theta = \{\theta_y, \theta_t, \theta_b\}$ wherein $\theta_y = (\beta^{'}, {\sigma^2}_{\varepsilon})$ is the unknown parameters of longitudinal sub-model, $\theta_t = ({\gamma_1}^{'}, ..., {\gamma_K}^{'}, \alpha^{'}, {\theta_{h_0}}^{'})$ is the vector of unknown parameters of competing risks sub-model and $\theta_b = vech(D)$ is the unknown parameters of the covariance matrix of random effects. To estimate the parameters of this joint model we can write the score vector of observed data as:

$$S(\theta) = \sum_i \frac{\partial}{\partial \theta^{'}} \log \int P(t_i, \delta_i | b_i; \theta_t, \beta) P(y_i | b_i; \theta_y) P(b_i; \theta_b) db_i \quad (5)$$

$$= \sum_i \int \frac{\partial}{\partial \theta^{'}} log\{P(t_i, \delta_i | b_i; \theta_t, \beta) P(y_i | b_i; \theta_y) P(b_i; \theta_b)\} P(b_i | t_i, \delta_i, y_i; \theta) db_i, \quad \delta_i = 1, 2, ..., K. \quad (6)$$

To calculate the maximum likelihood estimation of parameters in (5), the EM algorithm can be used to obtain parameter estimation from the expected value of the complete data log-likelihood at the rth iteration of

$$Q(\theta | \theta^{(r)}) = \sum_i \int \log\{P(t_i, \delta_i, y_i, b_i; \theta)\} P(b_i | t_i, \delta_i, y_i; \theta^{(r)}) db_i \quad (7)$$

$$= \sum_i \int \{\log P(t_i, \delta_i | b_i; \theta) + \log P(y_i | b_i; \theta) + \log P(b_i; \theta)\} P(b_i | t_i, \delta_i, y_i; \theta^{(r)}) db_i. \quad (8)$$

Therefore, it can be seen that obtaining estimates requires complex and long calculations due to multiple integrations in (7). Also, multiple integrations in this equation do not have a closed form. [3] proposed an approximation for parameter estimation and the expected value of the complete data log likelihood function. We use it in this investigation.

Let $\hat{\theta} = (\hat{\theta}_y^{'}, \hat{\theta}_t^{'}, \hat{\theta}_b^{'})'$ be the estimation of the full joint model parameters in (3) and $\tilde{\theta} = (\tilde{\theta}_y^{'}, \tilde{\theta}_b^{'})'$ be the estimator obtained from the linear mixed effect model in (1), then the expected function of the complete data log likelihood at $\hat{\theta}$ has the form of

$$
\begin{aligned}
E[\log\{P(t,\delta,y,b;\hat{\theta})\}] &\overset{P}{\to} \sum_i \int (\log P(t_i,\delta_i|b_i;\hat{\theta}_t,\hat{\theta}_y) \\
&+ \log P(y_i|b_i;\hat{\theta}_b) + \log P(b_i;\hat{\theta}_b)) \times P(b_i;\tilde{b}_i,\tilde{H}_i^{-1})db_i \qquad (9) \\
&\approx \sum_i \log P(t_i,\delta_i,\tilde{b}_i,\ \hat{\theta}_t,\tilde{\theta}_y) + \log P(y_i,\tilde{b}_i;\tilde{\theta}_y) + \log P(\tilde{b}_i;\tilde{\theta}_b),
\end{aligned}
$$

where $\tilde{b}_i = \arg\max_b \{\log P(y_i,b;\tilde{\theta}_y)\}$ and Hessian matrix,
$\tilde{H}_i^{-1} = (-\partial \log P(y_i|b,\tilde{\theta}_y))/(\partial b \partial b')|_{b=\tilde{b}_i}$ (See [3], for more details).

2.2 Two-stage approach for JMLC

Based on the two sub-models defined earlier, we implement the two-stage method for joint modeling of longitudinal and competing risks data as follows:

In the first stage, we fit the linear-mixed effects model for longitudinal process as described in (1) and the coefficients of the fixed effects (β), the covariance matrix D and the predictor of the random effects b_i can be estimated. Therefore, based on the observed longitudinal measurements, we can write the fitted longitudinal sub-model of (1) as follows:

$$
\begin{aligned}
\tilde{y}_i(s_{ij}) &= \tilde{m}_i(s_{ij}) + \varepsilon_i(s_{ij}) \qquad (10) \\
&= X_i^{'}(s_{ij})\tilde{\beta} + Z_i^{'}(s_{ij})\tilde{b}_i + \varepsilon_i(s_{ij}),
\end{aligned}
$$

where, $i = 1,\ldots,n$ and $j = 1,\ldots,n_i$.

In the second stage, we use the fitted values of parameters in the first step to estimate competing risks parameters, that is

$$h_{ik}(t) = h_{0k}(t)exp\{\gamma_k^{'} w_i + \alpha_k \tilde{m}_i(t)\},\ i = 1,...,n,\ k = 1,...,K, \quad (11)$$

where, $\tilde{m}_i(t)$ is estimated in the first stage and is considered as a covariate in the sub-model of the competing risks data. According to the approximation (9), we obtain the estimation of parameters in the competing risks process for each failure type by maximizing the following approximation term:

$$\sum_i \log P(t_i, \delta_i, \tilde{b}_i,\ \hat{\theta}_t, \tilde{\theta}_y) + \log P(y_i, \tilde{b}_i; \tilde{\theta}_y) + \log P(\tilde{b}_i; \tilde{\theta}_b),\ \delta_i = 1,2,...K.$$

Here, the density function of event process is given by

$$\begin{aligned} P(t_i, \delta_i, \tilde{b}_i,\ \tilde{\theta}_y; \theta_t) &= \prod_{k=1}^{K} h(t_i|\mathscr{M}_i(t_i), w_i, \tilde{\beta}; \theta_t)^{I(\delta_i=k)} S(t_i|\mathscr{M}_i(t_i), w_i, \tilde{\theta}_y; \theta_t) \\ &= \prod_{k=1}^{K} [h_0(t_i) \exp\{\gamma_k^{'} w_i + \alpha_k \tilde{m}_i(t_i)\}]^{I(\delta_i=k)} \\ &\times \exp\left(-\sum_{k=1}^{K} \int_0^{t_i} h_0(s) \exp\{\gamma_k^{'} w_i + \alpha_k \tilde{m}_i(s)\} ds \right). \end{aligned}$$

Also, the density function of longitudinal data given the random effects is given as follows:

$$\begin{aligned} P(y_i|\tilde{b}_i; \tilde{\theta}_y) P(\tilde{b}_i; \tilde{\theta}_b) &= \prod_{j=1}^{n_i} P\{yi(s_{ij})|\tilde{b}_i; \tilde{\theta}_y\} P(\tilde{b}_i; \tilde{\theta}_b) \quad (12) \\ &= \prod_{j=1}^{n_i} \frac{1}{(2\pi\tilde{\sigma}^2)^{(\frac{n_i}{2})}} \exp\left(-\frac{||yi(s_{ij}) - X_i^{'}(s_{ij})\tilde{\beta} + Z_i^{'}(s_{ij})\tilde{b}_i||^2}{2\tilde{\sigma}^2} \right) \\ &\times (2\pi)^{-(q_b/2)} det(\tilde{D})^{-1/2} \exp(-\tilde{b}_i^{'} \tilde{D}^{-1} \tilde{b}_i/2). \end{aligned}$$

2.2.1 Parameter estimation

Here, we present an algorithm to estimate the parameters of the joint modeling of longitudinal measurements and competing risks data based on a two-stage approach as follows:

- Stage I : By using a standard software of mixed-effects model, like the "lme" function, we estimate model (1) with all available observed longitudinal data and obtain $\tilde{\theta}_y$, $\tilde{\theta}_b$ and $\tilde{b}_i$.
- Stage II: In this stage, we use the conditional EM algorithm and estimate the parameters of the competing risks process based on the one-step Newton-Raphson method. We denote the vector of unknown parameters of the time-to-event process for each failure type as $\theta_{t_k} = (\theta_{h_{0k}}, \gamma_k, \alpha_k)$, $k = 1, ..., K$.
- Stage III: Perform the stage II for all failure types k, $k = 1, 2, ..., K$ and complete the competing risks process.

3 Application

One dataset used here to demonstrate the issues in competing risks analysis is the "standard and new anti-epileptic drugs" (SANAD) study which was a non-blinded randomized controlled trial enrolling patients with epilepsy to examine anti-epileptic drugs (AEDs). We could refer to [5] in order to see the published design and analysis of this trial. Here, the withdrawal of a randomized drug is considered as the time for treatment failure. Patients may decide, due to inadequate seizure control (ISC), to switch to another AED or to begin an additional AED. Also, patients may withdraw from a treatment because of an unacceptable adverse effect (UAE). We use the results from a competing risks analysis of the data for pairwise lamotrigine (LTG) versus carbamazepine (CBZ).

This data set includes 605 patients whom; CBZ (n=292) compares to LTG (n=313). 94 patients withdrew from the randomized drug because of UAE whereas 120 withdrew because of ISC within a maximum follow-up time of 6.6 years (median = 2.9 years). Withdrawals due to other reasons were considered as non-informative and patients were censored in these times. The maintenance dose recommended in the SANAD trial was independently considered reasonable and the approach to calibration sensible. Therefore, these calibrated doses are taken to be the longitudinal measurements within the competing risks joint model. On average, 4.6 longitudinal measurements were recorded for patients. However, there are measurements between 1 and 15 records.

We analyze these dataset using the proposed two-stage approach joint modeling of the previous sections. As defined earlier, calibrated dose can be considered as a response variable. We consider the following linear mixed effect model for the longitudinal data sub-model:

$$y_i(s_{ij}) = \beta_0 + \beta_1 s_{ij} + \beta_2 LTG_i + \beta_3 LTG_i s_{ij} + b_{i0} + b_{i1} s_{ij} + \varepsilon_{ij}, \qquad (13)$$

where, the LTG_i is a binary time-independent treatment effect that gives value 1 if patient i is randomized to LTG and zero if the patient is randomized to CBZ. $(b_{i0}, b_{i1})'$ and ε_{ij} are distributed as $N_2(0, D)$ and $N(0, \sigma_\varepsilon^2)$, respectively. Also a cause-specific hazard model for the competing risk data is defined as:

$$h_k(t_i) = \lambda_k exp\{LTG_i \gamma_k + \alpha_k m_i(t)\}, k = 1, 2, \qquad (14)$$

where, the baseline hazard function is supposed to have an exponential distribution. Under this formulation, parameters γ_1 and α_1 denote the effects of treatment (LTG) and dose calibrated, respectively, on the risk of ISCs and γ_2 and α_2 denote the same effects for UAEs. As an illustration, we also consider a separate model and a joint model ([1]) to compare with this proposed two-stage approach. We use the **lme**() function for the longitudinal measurement and the **coxph**() function in package **survival** for time-to-event process to fit separate models. Also the **joint Model()** function in **JM** package is used to fit the joint model.

A summary of parameter estimates and 95% confidence intervals (CIs) for the longitudinal sub-model and the competing risks sub-model parameters are given in Table 1. In this table, AIC (Akaike information criterion), BIC (Bayesian information criterion) and computation time are considered to be used for comparing models. Also the estimation of parameters for error term and the random effects covariance matrix are given in Table 2.

Table 1: Parameter estimates and 95% confidence intervals (CIs) for the longitudinal sub-model and the competing risks sub-model.

	Model parameter	Proposed two-stage	Separate	Joint model
Longitudinal Process	$\beta_0(Intercept)$	1.932	1.932	1.932
	$(95\%\ CI)$	(1.826,2.039)	(1.826,2.039)	(1.858,2.005)
	$\beta_1(Time)$	0.151	0.151	0.045
	$(95\%\ CI)$	(0.085,0.217)	(0.085,0.217)	(0.013,0.077)
	$\beta_2(Treat(LTG))$	-0.087	-0.087	0.041
	$(95\%\ CI)$	(-0.236,0.061)	(-0.236,0.061)	(-0.079,0.162)
	$\beta_3(Time:Treat)$	0.214	0.214	0.404
	$(95\%\ CI)$	(0.123,0.303)	(0.123,0.303)	(0.365,0.444)
Event Process	λ_{ISC}	0.240	-	-
	$(95\%\ CI)$	(0.020,0.028)	-	-
	λ_{UAE}	0.218	-	-
	$(95\%\ CI)$	(0.174,0.262)	-	-
	γ_{ISC}	0.011	0.015	-0.134
	$(95\%\ CI)$	(-0.222,0.244)	(-0.344,0.374)	(-0.497,0.230)
	γ_{UAE}	-0.460	-0.608	-0.614
	$(95\%\ CI)$	(-0.787,-0.133)	(-1.102,-0.192)	(-1.534,0.306)
	α_{ISC}	0.546	-	0.598
	$(95\%\ CI)$	(0.487,0.605)	-	(0.443,0.751)
	α_{UAE}	-0.503	-	-0.942
	$(95\%\ CI)$	(-0.615,-0.390)	-	(-1.455,-0.428)
Comparition Criteria	AIC	7225.824	8260.968	7210.406
	BIC	7234.771	8315.179	7342.563
	$Computation\ time$	1.33 min	<1 s	4.8 min

As the results of Table 1 indicate, the estimated treatment effect on calibrated dose, β_2, is non-significant for all models. The estimated of fixed effect for time, β_1, is significant for all models. That is to say, as time passes, the average calibrated dose increases. Also the estimated effect of treatment and time interaction, β_3, is significant for all models. Looking

Table 2: Parameter estimates of error term and the distinct components of the covariance matrix $\mathbf{D}$ (D_{11}, D_{12} and D_{22}).

parameter	Proposed two-stage	Separate	Joint model
σ	0.446	0.446	0.472
D_{11}	0.711	0.711	0.748
D_{22}	0.150	0.150	0.066
D_{12}	0.062	0.062	0.157

at the considered hazard model (14), we conclude that the overall treatment effect on the event hazard is divided into the direct effect γ_k and the indirect effect $\alpha_k(\beta_2 + \beta_3 t_{ij})$. Hence, the direct treatment effect must be considered by adjusting for treatment-specific intercept and slope of dose titration in the hazard model. The direct treatment effect on ISC, γ_{ISC}, is non-significant for all models in table 1. However, the direct treatment effect on UAE is significant for proposed two-stage joint modeling approach. Hence, if LTG is tested at the same CBZ rate, the useful effect of LTG on a UAE would still be evident and the difference in seizure control between the two drugs is unclear.

The estimation of association parameter, α_{ISC} and α_{UAE} is both significant for two considered models in table 1, which suggests that calibrated dose of drug is associated with time to treatment failure for both failure types.

The estimation of baseline hazard parameters, λ_{ISC} and λ_{UAE}, is both significant in the two-stage approach. Based on the given values of AIC, BIC and computation time we can see that the proposed model performs well and it is clear that BIC of proposed model is less than those of other models. AIC of proposed model is less than that of separate model although it is a little more than the joint model. However, given that the computation time is less than the computation time of the joint model, it can be said that the proposed model has performed well compared to the other models.

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Life Extension for Engineering Systems

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Abstract: The life extension for the engineering system consisting of independent components with an increasing failure rate functions is considered. The maintenance action is applied in a fixed component called the target component. This aim is also provided for the whole system. To this end, minimal repair and cold standby actions are regarded. We also consider two alternative policies for the target component. A component following a new random variable, and another following the same distributions of the target component. These policies obviously increase the reliability and life of the target component and consequently, the life and reliability of the engineering systems are also increased. In this regard, the life of the system is also extended. The optimality issues regarding the aforementioned statements are also described. Finally, some numerical results considering these life extensions are presented.

Keywords: Cold standby, Engineering system, Minimal repair, Preventive maintenance.

1 Introduction

The lifetime of engineering systems plays an important role for manufacturers and customers. Accordingly, the manufacture decides about guarantee, warranty, and price and the customers decide about payment values.

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Increasing the lifetime of any system results in benefits for both manufactures and customers. To this end, several strategies have been investigated by many scholars. In this study, we provide some methods of extending the lifetime of engineering systems called component-based and system-based ways.

Let $T = \psi(X_1, X_2, \ldots, X_n)$ denote the lifetime of engineering system consisting of independent components whose lifetimes are denoted by $X_1, X_2, \ldots, X_n$. Thus there exist a multinomial expression representing the reliability of such a system [10]. For more details about definitions, structure, relation, and ... of engineering system, we refer the reader to [6, chapter 4], [3] and [14] . The applications of engineering systems are very vast. In fact, many engineering systems, all of the k-out-of-n, series, and parallel systems with independent and identically distributed lifetimes of their components are excellent examples of engineering systems. Some practical applications of engineering systems are as follows:

I : Communications system [11].

II : Data processing systems [13].

III : Tires of a car.

IV : Series system.

V : Parallel system.

VI : k-out-of-n system.

VII : Series-Parallel system.

VIII : Parallel-Series system.

It is obvious that all customers are interested in reliable systems. Increasing the reliability of engineering systems is a favorite topic of many scholars. These methods were introduced by [3], and developed recently by [6, 11, 8, 2, 9, 5]. Among them, redundancy or maintenance actions can be enumerated. There exist many papers dealing with maintenance or redundancy theories for a special case of engineering systems like coherent, k-out-of-n, parallel, and series systems but in the case of just coherent systems, the study numbers are few.

The active redundancies for engineering systems and their dependent components are studied in [14],[15]. Navarro et al. [10], utilizing the copula functions, investigate three different policies of the minimal repair of failed components for a coherent system with dependent components. Optimal age replacement of a coherent system consisting of independent, identical, and increasing failure rate components are provided in [4]. The life of a class of coherent system consisting of independent and increasing failure rate is optimally extended in [7]. In this study, we consider engineering systems with independent and heterogeneous components. The results can obviously be used in the case of homogeneous components. The component's lifetime considered following distributions with increasing failure rate properties. In this class, the maintenance activities are applied for a fixed component. These activities involve the minimal repair (replaced the component when it fails with the same component at the same age), the perfect repair (replaced the component during its work with another component), and the cold standby (replaced the component when it fails with another component). The aim is to increase the reliability of the system and consequently increasing the mean to failure (MTTF) of the coherent systems.

The rest of the paper is organized as follows. In section 2, we present the formulas of minimal repair and cold standby activities. Furthermore, the MTTF of an engineering system is provided as our objective functions. Sections 3, and 4, deal respectively with life extension of engineering systems taking minimal repair, perfect repair, and cold standby maintenance. Finally, the conclusion of our study is presented in section 5.

2 Model description

Assume that X and Y be two non-negative independent variables respectively following absolutely continuous cumulative distribution functions (CDF) F and G. Thus the CDF of $X+Y$ (convolution) is given by ([10, 5]):

$$F*G(t) = 1-\bar{F}(t) - \int_0^t \bar{G}(t-x)f(x)dx, \tag{1}$$

where f indicates the probability distribution function (pdf) of the random variable X. The relation 1 can be used in the cold standby procedure of a component with CDF F by component with CDF G. Under a perfect repair in cold standby process, it is obvious that $F=G$, and consequently we have

$$F**G(t) = 1-\bar{F}(t) - \int_0^t \bar{F}(t-x)f(x)dx. \tag{2}$$

Regarding minimal repair policy, the failed component $X=x$ with CDF F is replaced by a worked component with the age of x following the CDF G. Hence the conditional revolution of $X+Y$ is as follows ([10, 5]):

$$F\#G(t) = 1-\bar{F}(t) - \int_0^t \frac{\bar{G}(t)}{\bar{G}(x)}f(x)dx, \tag{3}$$

and for the case of $F=G$ it is easy to see that:

$$\begin{aligned} F\#\#G(t) &= 1-\bar{F}(t) - \int_0^t \frac{\bar{F}(t)}{\bar{F}(x)}f(x)dx \\ &= 1-\bar{F}(t)+\bar{F}(t)\log\bar{F}(t). \end{aligned} \tag{4}$$

Here, assume that T represents the lifetime of a coherent system consisting of independent component lifetimes $X_1, X_2, \ldots, X_n$ following CDFs $F_1, F_2, \ldots, F_n$ respectively. Thus $F_T(t)$ is a multinomial expression of F_is [10, 3]. The MTTF of such a system is given by

$$\mu_t = \int_0^\infty [1-F_T(t)]dt \tag{5}$$

In this study, we are going to investigate the aforementioned policies on the MTTF of a coherent system. In continue without loss of generality, the number of components is considered as $n = 3$. Furthermore, some main situations are listed and their coefficients are derived.

system	structure	CDF($F_T(t)$)
I	$max(X_1, min(X_2, X_3))$	$F_1(t)(1-(1-F_1(t))(1-F_2(t)))$
II	$min(X_1, max(X_2, X_3))$	$1-(1-F_1(t))(1-F_2(t)F_3(t))$
III	$min(X_1, X_2, X_3)$	$1-(1-F_1(t))(1-F_2(t))(1-F_3(t))$
IV	$max(X_1, X_2, X_3)$	$F_1(t)F_2(t)F_3(t)$

Table 1: A coherent system with 3components

The most used statistical distribution in reliability analysis is Weibull distribution. This distribution has so important properties that are extensively discussed in the literature. Based on the Weibull distribution, there were constructed so many flexible modified distribution that can be utilized in reliability modeling. For a comprehensive discussion on Weibull distribution and it's modified see [12], [1]. The Weibull distribution with shape parameter α and scale parameter λ denoted by $W(\alpha,\lambda)$ is given by

$$f_X(x) = \frac{\alpha x^{\alpha-1}}{\lambda^\alpha}\exp(-(\frac{x}{\lambda})^\alpha).$$

If $\alpha > 1$ the distribution has an increasing failure rate feature and can be used for the modeling of components lifetime. In this study, we consider the scale parameter 1 and the shape parameter $1.5, 2, 2.5$ respectively for components $1, 2, 3$.

3 Minimal repair

Under a minimal repair policy, the failed component is replaced by another one having the same age. The alternative unit could have the same reliability with the failed component (4) or not (3). If T and μ_t denote the lifetime and MTTF of a coherent system, thus $F_T(t) = q(F_1(t), F_2(t), \ldots, F_n(t))$ where q is multinomial expression of CDFs of lifetime components $X_1, X_2, \ldots, X_n$. Moreover, it is directly seen that $\mu_t = \int_0^\infty [1 - F_T(t)]dt$. Now, consider a minimal repair action on the $i-$th component. The extended lifetime and extended MTTF (EMTTF) of the new version of this system is given by

$$F_T\#^i G(t) = q(F_1(t), F_2(t), \ldots, F_i\#G(t), \ldots, F_n(t)),$$

and

$$\mu_t\#^i G = \int_0^\infty [1 - F\#^i G_T(t)]dt.$$

and similarly

$$F_T\#\#^i F_i(t) = q(F_1(t), F_2(t), \ldots, F_i\#G(t), \ldots, F_n(t)),$$

and

$$\mu_t\#\#^i F_i = \int_0^\infty [1 - F\#^i G_T(t)]dt.$$

Minimal repair	Component 1	Component 2	Component 3
μ_t	1.050	1.050	1.050
Same	1.540	1.123	1.118
$W(1.5,2)$	2.314	1.155	1.165
$W(2,2)$	2.109	1.158	1.166
$W(2.5,2)$	2.019	1.160	1.169
$W(3,2)$	1.974	1.163	1.171

Table 2: The EMTTF of the system I under minimal repair policy

Minimal repair	Component 1	Component 2	Component 3
μ_t	0.707	0.707	0.707
Same	0.968	0.781	0.775
$W(1.5,2)$	1.053	0.846	0.848
$W(2,2)$	1.063	0.846	0.848
$W(2.5,2)$	1.072	0.848	0.849
$W(3,2)$	1.080	0.850	0.851

Table 3: The EMTTF of the system II under minimal repair policy

The numerical results due to the minimal repair policies for systems I and II are tabulated in Tables 2 and 3. The optimal EMTTF of corresponding systems is also represented. These values are obviously present the good performances of the minimal policy.

4 Cold standby

Under a cold standby process, the failed component is replaced by a new one. The alternative unit could have the same reliability with the failed component (2) or not (1). If T and μ_t denote the lifetime and MTTF of a coherent system, thus $F_T(t) = q(F_1(t), F_2(t), \dots, F_n(t))$ where q is multinomial expression of CDFs of lifetime components $X_1, X_2, \dots, X_n$. Moreover, it is directly seen that $\mu_t = \int_0^\infty [1 - F_T(t)]dt$. Now, consider a minimal repair action on the $i-$th component. The extended lifetime and extended MTTF (EMTTF) of the new version of this system is given by

$$F_T *^i G(t) = q(F_1(t), F_2(t), \dots, F_i * G(t), \dots, F_n(t)),$$

and

$$\mu_t *^i G = \int_0^\infty [1 - F *^i G_T(t)]dt.$$

and similarly

$$F_T **^i F_i(t) = q(F_1(t), F_2(t), \dots, F_i ** G(t), \dots, F_n(t)),$$

and

$$\mu_t **^i F_i = \int_0^\infty [1 - F **^i G_T(t)]dt.$$

The cold standby policy performing for systems III and IV in the numerical form are tabulated in Tables 4 and 5. The MTTF of these systems is extended in optimal form. The corresponding values are also provided.

5 Conclusion

An engineering system consisting of independent and repairable components are considered. The lifetime of their components is considered with

Minimal repair	Component 1	Component 2	Component 3
μ_t	0.500	0.500	0.500
Same	0.624	0.600	0.586
$W(1.5,2)$	0.638	0.606	0.589
$W(2,2)$	0.643	0.609	0.591
$W(2.5,2)$	0.645	0.611	0.593
$W(3,2)$	0.646	0.611	0.593

Table 4: The EMTTF of the system III under cold standby process

Minimal repair	Component 1	Component 2	Component 3
μ_t	1.320	1.320	1.320
Same	1.914	1.882	1.874
$W(1.5,2)$	2.756	2.749	2.751
$W(2,2)$	2.706	2.699	2.700
$W(2.5,2)$	2.698	2.690	2.692
$W(3,2)$	2.703	2.695	2.696

Table 5: The EMTTF of the system IV under cold standby process

increasing failure rate features. Three replacement policies including cold standby and minimal repair policies applying to these components are investigated. The aim of performing these actions is to extend the life of the system. The main question is when the operator acts preventive maintenance to achieve optimal EMTTF of the systems. The corresponding relations of these policies have been studied and the comparison between these policies is tabulated through some simulation studies.

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Maintenance in Presence of Incomplete Data in Series Systems

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Abstract: Maintenance optimization problems have received much attention recently. Applying optimization actions in different contexts leads to encounter expected and unexpected challenges. Inspections and monitoring make useful data for optimizing maintenance policies and collected data play an especial role in such problems. But sometimes data are incomplete. In this paper, masking will be discussed in terms of the incompleteness of data.

Keywords: Incomplete data, Maintenance optimization, Masking.

1 Introduction

Today maintenance actions are an indispensable part of life because of excessive use of machines and systems in our life. The increasing leap in science made a huge change in human life and has made life dependent on different machines and systems such that life without machines and systems is approximately impossible. Since performing any maintenance action any time encountered some restrictions such as time and cost,

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different types of maintenance have emerged.
Generally, maintenance actions are classified into two categories: corrective maintenance (CM) where maintenance activities are carried out when the system is failed and preventive maintenance (PM) where maintenance activities are performed when the system is operating. Also, there are different types of preventive and corrective maintenance, for more details one is referred to [1] and references therein.

Another class of maintenance actions is called opportunistic maintenance (OM). Cavalcante and Lopes defined OM as a systematic method of collection, investigation, and preplanning activities for generating a set of maintenance tasks to act on in the occurrence of an opportunity [2].

In the reliability analysis of series systems, time to failure and the exact cause of failure (complete data) are collected in order to do different statistical analysis such as estimation of the reliability function and maintenance modeling. But, sometimes collected data are incomplete. Here we assumed that the exact cause of failure is unidentifiable (because of improper diagnostic equipment storage, or time and cost restrictions) and it is only known that the exact cause of failure belongs to a minimum random subset(MRS) of all possible causes. These data are called to be masked [3, 4].

In this paper, an opportunity based perfect preventive maintenance policy is considered in presence of incomplete data like masking. A perfect preventive maintenance action restores an operating deteriorated (non-failed) component to an as good as new (AGAN) state, for instance replacing it by a new one. Here, the maintenance policy is designed such that the perfect preventive replacement is opportunistic since the system fails besides the failed component, some endangered components are replaced by a new one. Also, inspections are applied periodically.

The rest of the paper is organized as follows. In section 2, the model is explained. In section 3, the maintenance model is proposed. The average long-run maintenance cost is presented in section 4. A numerical example is conducted in order to illustrate the applicability of the proposed method in section 5. Finally, the conclusion is given in section 6.

2 Model Description

Suppose that we have a series system with J components that operates in a static environment. Moreover, we suppose that when the system fails we observe failure time, t, but the exact cause of failure might be unknown, and we only know that it belongs to MRS of $\{1,2,...,J\}$. Let M be the observed MRS corresponding to the failure time t for the system. The set M essentially includes components that are possible to be cause for system failure and if $M = \{1,...,J\}$ then the system cause of failure is called to be completely masked. Thus the known information is given as follows:

$$(t, M). \tag{1}$$

To obtain the reliability function of the system and propose its maintenance modeling some assumptions have been made as follows:

1. Let $T_l;\ l = 1,2,...,J$ be the lifetimes of the lth component (independent components) and assume that the system fails only due to one of the J components, therefore the system failure time (T) is defined to be $T = min(T_1,...,T_J)$.
2. T_l belongs to a continuous distribution family with probability density and reliability functions denoted by $f_l(t)$ and $R_l(t)$, respectively.
3. The reliability function of T is given by

$$R(t) = R(t;\theta) = \mathbb{P}_\theta(T > t) = \prod_{l=1}^{J}[1 - F_l(t)] \tag{2}$$

   where $\theta = (\theta_1,...,\theta_J)$, θ_l is the set of parameters related to the lth component and F_l is corresponding distribution function of lth component.
4. Suppose K be a random variable which indicates the cause of failure for the system. Then the joint probability density function of (T,K) is

given by

$$f_{T,K}(t,l) = f_l(t) \prod_{j \neq l} [1 - F_j(t)] \tag{3}$$

where, the joint distribution of T and K can be specified in terms of the so called sub-distribution function $F(j,t) = \mathbb{P}(K = j, T \leq t)$, or equivalently by the sub-reliability function $R(j,t) = \mathbb{P}(K = j, T > t)$ [5].

5. $p_l^t(M_i) = \mathbb{P}(M = M_i | T = t, K = l)$ is called the masking probability, where M_i is an observation of M. Some authors such as Mukhopadhyay [6], Kuo and Yang [7] and Cai and et al. [8], assumed

$$\mathbb{P}(M = M_i | T = t, K = l) = \mathbb{P}(M = M_i | K = l) = p_l(M_i),$$

that is, the masking probability is independent of failure time, but is dependent to the causes of failure. We assume similar to a new approach that was presented to model the dependency of the masking probability on the failure time and its exact cause using the multinomial logistic regression model [9].

6. Some constraints are considered for conditional masking probabilities. Suppose M be the set of all nonempty subsets of $\{1, ..., J\}$ that have $2^J - 1$ members. For $l = 1, ..., J$, define $M_l = \{M \in M : l \in M, l \in \{1, ..., J\}\}$ thus

$$\mathbb{P}_l^t(M_i) = \mathbb{P}(M = M_i | K = l, T = t) = 0 \qquad \forall M_i \in M_l^c = M - M_l$$

and

$$\sum_{M_i \in M} \mathbb{P}_l^t(M_i) = \sum_{M_i \in M_l} \mathbb{P}_l^t(M_i) = 1, l = 1, ..., J \tag{4}$$

denote $\mathbf{P}_l = \{\mathbb{P}_l^t(M_i) : M_i \in M_l\}, l = 1, 2, ..., J$ then the set of all masking probabilities is $\mathbb{P} = (\mathbb{P}_1, ..., \mathbb{P}_J)$.

3 Maintenance Modeling

In this section, a perfect preventive maintenance (PPM) policy is presented based on an opportunistic action and an optimal maintenance policy is derived using long-run cost rate criteria for a series system with J components. Inspections are assumed to be periodically at times $k\tau;\ \ k=1,2,...,$ with cost c_{ins} for the system and $M_k;\ \ k=1,2,...$ are corresponding masked sets. The time interval $((k-1)\tau,k\tau]$ is called the kth period. Maintenance actions are applied based on some assumptions including:

- Inspection is performed at the end of each period
- Time needed for inspections and maintenance actions is negligible
- The system failure is not self-announced
- Components are maintained independently

At kth inspection time, $k\tau$, a maintenance action is performed if the system has been failed during $((k-1)\tau,k\tau]$ interval, that is,

$$T>(k-1)\tau \quad \& \quad T<k\tau.$$

Since it is assumed that exact cause of failure is unknown and it belongs to possibly masked set, $M_k\subseteq\{1,2,...,J\}$, thus the probability of each cause in M_k given possibly masked set and interval censored failure time is given by

$$p_{jM_k}=\mathbb{P}(K=j|M_k,u\in((k-1)\tau,k\tau])=\frac{\int_{(k-1)\tau}^{k\tau}\mathbb{P}(M_k|j,u)f_{T,K}(u,j)du}{\int_{(k-1)\tau}^{k\tau}\sum_{l'\in M_k}\mathbb{P}(M_k|l',u)f_{T,K}(u,l')du} \tag{5}$$

where u is the exact failure time. Note that $p_{jM_k}=0$ for $j\notin M_k$.

Eventually, when the system is failed at $((k-1)\tau,k\tau]$ a maintenance action is carried out for each component in M_k according to a predetermined value of $\rho;\ \ 0<\rho<1$, as follows:

- If $T_l > (k-1)\tau \quad \& \quad T_l < k\tau$ then perfect corrective maintenance (PCM) action is done for component l with cost c_{lc} and probability $P_{cl}(k\tau)$ (that is, the failed component l is replaced by a new one).
- If $T_l > k\tau \quad \& \quad p_{lM_k} > \rho$ then opportunistic perfect preventive maintenance (OPPM) action is done for component l with cost $c_{lp} < c_{lc}$ and probability $P_{pl}(k\tau)$ (that is, the degraded component l is replaced by a new one).

Otherwise, no maintenance action is done.

4 Long-run cost rate

The time from the component installation to its first replacement or the time between two successive replacement of each component is referred to as a renewal cycle. Let L and L_j denote the average long-run maintenance cost per unit of time for the system and component j, respectively. Therefore, based on the renewal reward theorem the expected long-run maintenance cost rate for component j is

$$L_j(\tau,\rho) = \lim_{t\to\infty}\frac{C_j(t)}{t} = \frac{E(C_{rj})}{E(T_{rj})} \tag{6}$$

where $E(C_{rj})$ and $E(T_{rj})$ are total expected cost during a replacement cycle and expected length of the replacement cycle for component j, respectively such that

$$E(C_{rj}) = \sum_{k=1}^{\infty}[(\frac{kc_{ins}}{J}+c_{jp})P_{pj}(k\tau)+(\frac{kc_{ins}}{J}+c_{jc})P_{cj}(k\tau)] \tag{7}$$

and

$$E(T_{rj}) = \sum_{k=1}^{\infty}k\tau[P_{pj}(k\tau)+P_{cj}(k\tau)]. \tag{8}$$

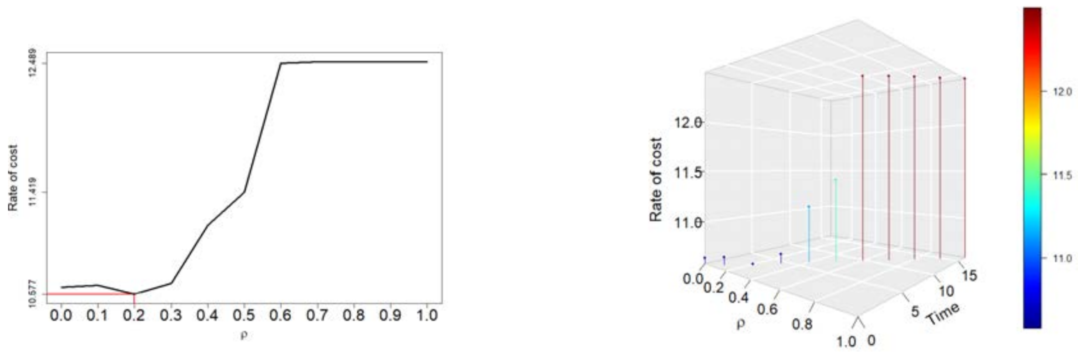

Figure 1: The optimal value of ρ as decision parameter considering $(\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.25, 0.7)$ and $(\beta_1, \beta_2, \beta_3) = (1.5, 1.25, 1.75)$

Finally, the total expected long-run maintenance cost rate for the series system until time t is given by (see [10])

$$L(\tau, \rho) = \sum_{j=1}^{J} L_j(\tau, \rho). \tag{9}$$

5 Numerical Example

In this section, an example illustrates the proposed method numerically. We assume that a series system has $J = 3$ components that follow Weibull distribution with parameters $(\alpha_1, \beta_1) = (0.5, 1.5)$, $(\alpha_2, \beta_2) = (0.25, 1.25)$ and, $(\alpha_3, \beta_3) = (0.7, 1.75)$.
The system is monitored periodically at times $k\tau$; $k = 1, 2, \ldots$ and we assume $\tau = 0.88$.

According to the average long-run cost rate criteria, optimal value of ρ was derived and depicted at Figure 1 based on the equation 9.

6 Conclusion

In this paper, maintenance optimization is investigated in presence of incomplete data like masking. An opportunistic perfect preventive maintenance policy is considered to handle masked data. A series system with 3 components is considered to study the proposed method numerically. The lifetime of components follows the Weibull distribution. As discussed in the previous section, the proposed method is justified numerically and the optimal value of ρ as a threshold for applying opportunistic perfect preventive action is derived based on the average long-run cost rate criteria.

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Nonparametric Estimation of Copula Based Stress-strength Models

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Abstract: The purpose of this paper is to provide a nonparametric method for the estimation of copula-based stress-strength models. These method is based on improved probit transformation method for copula density estimation. This nonparametric method is a novel application based on an existing bivariate kernel method combined with Monte Carlo estimation without specification of the copula or the marginal distributions. Simulation results suggests that the nonparametric estimation method has better performance than the empirical esimation method.

Keywords: Copula, Probit transformation, Stress-strength.

1 Introduction

In reliability analysis, the stress-strength model describes the reliability of an individual which has a random strength X and is subject to a random stress Y. The individual fails if the strength cannot resist on the stress.

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Hence, $R = P(Y < X)$ represents the reliability of the individual. the stress-strength models have been widely discussed in the statistical and reliability literature. There are many literatures have investigated the stress-strength models under different distributions. It is usually assumed that the stress and strength variables are independent, then based on this assumption to analysis the characteristics of the stress-strength models. However, in many cases, the stress and strength variables are dependent in some way. Nevertheless, a bivariate distribution model needs the marginal distribution are the same type. To overcome this limitation, a copula-based approach, which admits the margins are any type and not necessarily belonging to the same family, was considered by some researchers. The Farlie-Gumbel-Morgenstern copula is used by [3] to analyze the dependence in stress-strength models. Recently, parametric and nonparametric inference for the reliability of copula-based stress-strength models is discussed by [2].

This paper provides a general framework for estimating the reliability in copula-based stress-strength models with an emphasis on model-robust inference. These method is based on improved probit transformation method for copula density estimation. We have not focused on any specific family of distributions (margins) or copulas.

The rest of the paper is arranged as follows. In Section 2, the preliminaries for copulas are described. The estimation of the copula density function using local likelihood probit transformation method is provided in Section 3. In Section 4, the nonparametric method for the estimation of copula-based stress-strength is presented. The simulation results are provided in Section 5 and concluding remarks are given in Section 6.

2 Copulas

Some definitions related to a copula functions will be briefly reviewed based on [5]. Let (X,Y) be a continuous random variable with joint cumulative distribution function (cdf) F, then copula C corresponding to F defined as:

$$F(x,y) = C(F_X(x), F_Y(y)), \qquad (x,y) \in R^2, \tag{1}$$

where F_X and F_Y are the marginal distributions of X and Y, respectively. A bivariate copula function C is a cumulative distribution function of random vector (U,V), defined on the unit square $[0,1]^2$, with uniform marginal distributions as $U = F_X(X)$ and $V = F_Y(Y)$.

The authors shall write $C(u,v;\theta)$ for a family of copulas indexed by the parameter θ. If $C(u,v;\theta)$ is an absolutely continuous copula distribution on $[0,1]^2$, then its density function is $c(u,v;\theta) = \frac{\partial^2 C(u,v;\theta)}{\partial u \partial v}$. As a result, the relationship between the copula density function (c) and the joint density function $f_{X,Y}(\cdot,\cdot)$ of (X,Y) according to equation (1) can be represented as

$$f_{X,Y}(x,y) = c(F_X(x), F_Y(y);\theta) f_X(x) f_Y(y), \qquad (x,y) \in R^2, \tag{2}$$

where $f_X(\cdot)$ and $f_Y(\cdot)$ are the marginal density functions of X and Y, respectively.

Table 1 presents summary information of some well-known bivariate copulas such as the parameter space and Kendall's tau (τ) of them. In this table, Clayton, Gumbel, and Frank copulas belong to the class of Archimedean copulas and Gaussian and T copulas belong to the class of Elliptical copulas. The copula-based Kendall's tau association for continuous variables X and Y with copula C is given by $\tau = 4\int_{[0,1]^2} C(u,v)dC(u,v) - 1$.

Table 1: Some well-known bivariate copulas

Copula	$C(u,v;\theta)$	Parameter Space	Kendall's τ
Clayton	$(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$	$\theta \in [0,+\infty)$	$\frac{\theta}{\theta+2}$
Gumbel	$\exp\left\{-\left[(-\ln u)^{\theta}+(-\ln v)^{\theta}\right]^{1/\theta}\right\}$	$\theta \in [1,+\infty)$	$\frac{\theta-1}{\theta}$
Gaussian[1]	$\Phi_2(\Phi^{-1}(u),\Phi^{-1}(v);\theta)$	$\theta \in [-1,+1]$	$\frac{2}{\pi} arcsin(\theta)$

3 Local likelihood probit transformation estimation

Transformation method to kernel copula density estimation was introduced by [1]. The simple idea is to transform the data so that it is supported on the full R^2. On this transformed domain, standard kernel techniques can be used to estimate the density. An adequate back-transformation then yields an estimate of the copula density.

Let $(U_i, V_i)_{i=1,\ldots,n}$ are independent and identically distributed observations from the bivariate copula C and the purpose is to estimate the corresponding copula density function. Denote Φ as the standard Gaussian distribution and ϕ as its first order derivative. Then $(S_i, T_i) = (\Phi^{-1}(U_i), \Phi^{-1}(V_i))$ is a random vector with Gaussian margins and copula C. According to (2), the corresponding density function can be written as $f(s,t) = c(\Phi(s), \Phi(t))\phi(s)\phi(t)$. Thus, an estimation of the copula density function can be given by

$$\hat{c}_n^{(\mathscr{P}\mathscr{T})}(u,v) = \frac{\hat{f}_n(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}, \qquad (u,v) \in (0,1)^2. \tag{3}$$

As the (U_i, V_i) are unavailable and one has to use the pseudo-transformed sample $(\hat{S}_i, \hat{T}_i) = (\Phi^{-1}(\hat{U}_i), \Phi^{-1}(\hat{V}_i))$, instead. As a first natural idea, the standard kernel density estimator for $\hat{f}_n$ in (3) can be considered as follows:

$$\hat{f}_n(s,t) = \frac{1}{n|\mathbf{H}_{ST}|^{\frac{1}{2}}} \sum_{i=1}^{n} \mathbf{K}\left(\mathbf{H}_{ST}^{-\frac{1}{2}} \begin{pmatrix} s - \hat{S}_i \\ t - \hat{T}_i \end{pmatrix}\right),$$

where $\mathbf{K} : R^2 \to R$ is a kernel function, and $\mathbf{H}_{ST} = b_n I$ is a bandwidth matrix.

This kernel estimator has asymptotic problems at the edges of the distribution support. To remedy this problem, local likelihood probit transformation ($\mathscr{L}\mathscr{L}\mathscr{P}\mathscr{T}$) method was recently suggested by [4]. Instead of applying the standard kernel estimator, they locally fit a polynomial to the

[1] Φ^{-1} is the inverse of the standardized univariate Gaussian distribution and Φ_2 is the standardized bivariate Gaussian distribution with correlation parameter θ.

log-density of the transformed sample. This method can fix the boundary issues in a natural way and able to cope with unbounded copula densities. Recently, [6] with a comprehensive simulation study has shown that $\mathscr{LLPT}$ method for copula density estimation yields very good.

Around $(s,t) \in R^2$ and (s',t') close to (s,t), the local log-quadratic likelihood estimation of $\log f(s,t)$ from the pseudo-transformed sample is defined as:

$$\begin{aligned} logf(s',t') &= a_{2,0}(s,t) + a_{2,1}(s,t)(s'-s) + a_{2,2}(s,t)(t'-t) \\ &\quad + a_{2,3}(s,t)(s'-s)^2 + a_{2,4}(s,t)(t'-t)^2 + a_{2,5}(s,t)(s'-s)(t'-t) \\ &\equiv P_{a_2}(s'-s, t'-t). \end{aligned}$$

The vector $a_2(s,t) \equiv (a_{2,0}(s,t), \cdots, a_{2,5}(s,t))$ is then estimated by

$$\begin{aligned} \hat{a}_2(s,t) = arg \max_{a_2} \Big\{ & \sum_{i=1}^{n} \mathbf{K}\Big(\mathbf{H}_{ST}^{-\frac{1}{2}} \binom{s-\hat{S}_i}{t-\hat{T}_i}\Big) P_{a_2}(\hat{S}_i - s, \hat{T}_i - t) \\ & - n \int_{R^2} \mathbf{K}\Big(\mathbf{H}_{ST}^{-\frac{1}{2}} \binom{s-s'}{t-t'}\Big) exp(P_{a_2}(s'-s, t'-t)) ds' dt' \Big\}. \end{aligned}$$

Therefore, the estimation of $f(s,t)$ is $\tilde{f}^p(s,t) = \exp\{\hat{a}_2(s,t)\}$ and thus $\mathscr{LLPT}$ estimator of a copula density is

$$\hat{c}_n^{(\mathscr{LLPT})}(u,v) = \frac{\tilde{f}^p(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}, \qquad (u,v) \in [0,1]^2. \quad (4)$$

When the underlying density is on $[0,1]^2$, the performance of the kernel estimator depends on the choice of the kernel function and the bandwidth (smoothing parameter). For bandwidth choice, a practical approach is to consider the minimization of the AMISE on the level of the transformed data. In this article, the bandwidth choice based on nearest-neighbor method; see [4].

4 Estimation of copula based stress-strength

It is natural to consider nonparametric methods especially when the data analyst is unsure about the specification of margins and copula. In this section, we propose a combination of Monte Carlo and bivariate kernel copula density estimation to obtain a nonparametric estimate of the reliability.

Let $\{(X_i, Y_i)\}_{i=1,\cdots,n}$ be a random sample of size n from dependent variabels X and Y. The empirical estimator of R based on these observations is

$$\hat{R} = \frac{1}{n}\sum_{i=1}^{n} I_{[Y_i<X_i]}. \tag{5}$$

The reliability for dependent X and Y can be rewritten based on copula density as

$$\begin{aligned} R = P(Y < X) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{x} f_{X,Y}(x,y)dydx \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{x} c(F_X(x), F_Y(y); \theta) f_X(x) f_Y(y)dydx. \end{aligned}$$

By considering $U = F_X(X)$ and $V = F_Y(V)$,

$$R = \int_0^1 \int_0^{F_X^{-1}(u)} c(u,v;\theta)dvdu. \tag{6}$$

Thus, the Kernel estimation of R can be presented as

$$\tilde{R} = \int_0^1 \int_0^u \hat{c}_n^{(\mathscr{LLPT})}(u,v)dvdu, \tag{7}$$

where $\hat{c}_n^{(\mathscr{LLPT})}(\cdot,\cdot)$ is the local likelihood probit transformation estimation of copula density in equation (4).

We summarize the steps for the construction of our proposed nonparametric estimator $\tilde{R}$:

1. Given the data (X_i, Y_i) obtain the pseudo observations $(\tilde{U}_i, \tilde{V}_i)$, where $\tilde{U}_i = R_i/(n+1)$, $\tilde{V}_i = S_i/(n+1)$ for $i = 1, \cdots, n$, and R_i and S_i are the ranks of the observation X_i and Y_i, respectively.

2. Build the estimated copula density with local likelihood probit transformation ($\mathscr{L}\mathscr{L}\mathscr{P}\mathscr{T}$) method.
3. Estimate the reliability by Monte Carlo method according to (7).

5 Simulation study

In this section, a Monto Carlo simulation is presented to illustrate the estimation methods which are described. We demonstrate that the suggested nonoparametric estimator based on local likelihood probit transformation method is efficient than the empirical estimator of R.

Consider, the dependent data (U,V) come from the Clayton, Gumbel and Gaussian copulas with Kendall's tau 0.2, 0.5, and 0.8 that are presented in Table 1. These copulas cover different dependence structures. Gaussian copula exhibit symmetric and no tail dependence in both lower and upper tails. The Clayton copula exhibits strong left tail dependence and the Gumbel copula has strong right tail dependence. Moreover, 1000 Monte Carlo samples of sizes $n = 100$ and 500 are generated from each type of copulas with maginals normal and exponential (by rate 1 and 2) distributions. The estimators obtained are compared via the Bias and root mean square error (RMSE).

Results of the simulation study are presented in Tables 2, 3, and 4. These tables present the Bias and RMSE relative to the two estimators of the respective copulas for different values of sample sizes and Kendall's tau and different marginal distributions. The simulation procedure was performed for the positive and negative values of Kendall's tau and according to the symmetry of the obtained results, the results have been reported only for positive values of Kendall's tau. As the results for the sample sizes greater than 500 were in line with our expectation that the increase in sample size will improve the parameter estimation, the corresponding results were omitted from the tables for brevity.

Table 2: Estimated Bias and RMSE of the empirical and kernel estimations for clayton copula

Margianls (X & Y)	n	τ	Empirical esimation		Kernel estimation	
			Bias	RMSE	Bias	RMSE
Normal & Normal	100	0.2	0.5170	0.5269	-0.0150	0.0126
		0.5	0.5229	0.5334	-0.0198	0.0158
		0.8	0.5346	0.5445	-0.0229	0.0186
	500	0.2	0.5032	0.5052	-0.0112	0.0114
		0.5	0.5117	0.5137	-0.0134	0.0103
		0.8	0.5286	0.5303	-0.0182	0.0093
Normal & Exp(2)	100	0.2	0.3079	0.3241	0.1915	0.1134
		0.5	0.3580	0.3710	0.2333	0.1735
		0.8	0.3885	0.4528	0.2704	0.2504
	500	0.2	0.2643	0.2872	0.1623	0.0923
		0.5	0.2946	0.3270	0.2050	0.1350
		0.8	0.3169	0.3997	0.2326	0.1823
Exp(1) & Exp(2)	100	0.2	0.7064	0.7118	-0.2932	0.2012
		0.5	0.8113	0.8151	-0.3602	0.3090
		0.8	0.8610	0.8640	-0.4123	0.3639
	500	0.2	0.6287	0.6399	-0.2016	0.1824
		0.5	0.7051	0.7258	-0.3076	0.2648
		0.8	0.7602	0.7507	-0.3620	0.3285

The results show that estimated Bias and RMSE of estimations decrease as sample size increases and estimations improve. The accuracy of the estimations decrease with increasing Kendall's tau. Based on Bias and RMSE, the results show that the empirical esimation ($\hat{R}$) has better performance than the kernel estimation ($\tilde{R}$). Finally, it is necessary to note that although the time required to compute the kernel estimation method is longer than the empirical esimation method, but the kernel estimation method has accurate and acceptable results especially for normal marginal distributions.

6 Conclusions

In this paper, we studied the nonparametric estimation of the stress-strength reliability for dependent stress and strength variables based on

Table 3: Estimated Bias and RMSE of the empirical and kernel estimations for gumbel copula

Margianls (X & Y)	n	τ	Empirical esimation		Kernel estimation	
			Bias	RMSE	Bias	RMSE
Normal & Normal	100	0.2	0.4315	0.4117	0.0182	0.0120
		0.5	0.4911	0.5029	0.0236	0.0193
		0.8	0.5496	0.5684	0.0293	0.0226
	500	0.2	0.4015	0.3638	0.0142	0.0101
		0.5	0.4273	0.4393	0.0180	0.0163
		0.8	0.4989	0.5023	0.0211	0.0192
Normal & Exp(2)	100	0.2	0.4063	0.3201	0.1947	0.1154
		0.5	0.4655	0.3808	0.2380	0.1862
		0.8	0.5370	0.4516	0.2929	0.2561
	500	0.2	0.3016	0.2845	0.1641	0.0832
		0.5	0.3666	0.3499	0.2176	0.1129
		0.8	0.4487	0.4123	0.2532	0.1985
Exp(1) & Exp(2)	100	0.2	0.7130	0.5189	-0.2263	0.1812
		0.5	0.7933	0.5970	-0.2955	0.2364
		0.8	0.8280	0.6312	-0.3443	0.2978
	500	0.2	0.6561	0.4072	-0.2069	0.1445
		0.5	0.7046	0.4654	-0.2559	0.2012
		0.8	0.7687	0.5294	-0.3026	0.2495

copulas. The simulation results suggests that the nonparametric estimation of copula based stress-strength models via local likelihood probit transformation method has better performance than the empirical esimation method especially for normal marginal distributions.

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Table 4: Estimated Bias and RMSE of the empirical and kernel estimations for gaussian copula

Margianls (X & Y)	n	τ	Empirical esimation		Kernel estimation	
			Bias	RMSE	Bias	RMSE
Normal & Normal	100	0.2	0.6056	0.5166	0.0093	0.0114
		0.5	0.6402	0.5423	0.0131	0.0184
		0.8	0.6968	0.6060	0.0181	0.0223
	500	0.2	0.5445	0.4366	0.0061	0.0084
		0.5	0.5858	0.4673	0.01134	0.0124
		0.8	0.6231	0.5052	0.01572	0.0194
Normal & Exp(2)	100	0.2	0.3044	0.3177	0.1930	0.1831
		0.5	0.3505	0.3643	0.2364	0.2134
		0.8	0.4360	0.4536	0.2834	0.2535
	500	0.2	0.2070	0.2396	0.1629	0.1525
		0.5	0.2607	0.2938	0.2168	0.1961
		0.8	0.3450	0.3476	0.2534	0.2233
Exp(1) & Exp(2)	100	0.2	0.6966	0.7025	-0.2281	0.1923
		0.5	0.7926	0.7967	-0.2842	0.2375
		0.8	0.8339	0.8307	-0.3337	0.2943
	500	0.2	0.6194	0.6407	-0.1980	0.1632
		0.5	0.7162	0.6872	-0.2441	0.2045
		0.8	0.7817	0.7133	-0.3032	0.2564

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Optimal Replacement Time for a Parallel System with the Random Number of Dependent and Heterogeneous Components

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Abstract: The study is concerned with finding an optimal replacement time for a parallel system that consists of some random number of dependent and heterogeneous components. For dependence structure, we assume the copula model between the lifetime of corresponding components. In contrast to the previous works, the number of components is randomly supposed. The impact of this assumption is investigated. In particular, we numerically examine how the dependence between the components which are randomly distributed, affects the optimal replacement time for the system which minimize its mean cost rate function. We consider some different cases for the lifetime of components, including independent and identical, independent and not identical, dependent and identical, and finally dependent and not identical. In addition, the number of components is randomly assumed following an arbitrary positive and discrete probability mass function. In numerical results, the Poisson distribution is considered modelling the number of components. Comparative numerical results are presented for particularly chosen dependence models.

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Keywords: Copula, Heterogeneous, Parallel system, Random number of components.

1 Introduction

The assumption of independence of components in a system is rarely valid in practice. In most real life situations, the lifetimes of components are dependent. The components in a system may share the same load or may be subject to the same set of stresses. This will cause the lifetimes of components to be related to each other, or to be dependent [6]. In addition, circumstances may arise that the number of components in a system is not fixed. The random number of these components has effects on the reliability characteristics of a system. Reliability evaluation of systems consisting of dependent components has attracted a great deal of attention. Some recent works on systems with dependent components are in [15, 2, 10, 13, 14]. Other works that deal with a system consisting of a random number of components are investigated in [3, 12, 1, 5, 8, 9, 11]. Nakagawa [7] formulated two optimization problems to find the optimal value of the number of elements and optimal replacement time in a binary k-out-of-n system. A parallel system work when all of their components function. The optimal number of elements and optimal replacement time have been analytically obtained by [7] when the system consists of independent components each having the same exponential lifetime distribution. Nakagawa and Zhao [8] considered the same optimization problems for a parallel system with a random number of units when the components are independent. Eryilmaz [3] studied the optimal number of units and optimal replacement time for a parallel system consisting of a random number of units when the distribution of the number of units follows a power series class of distributions. Recently Eryilmaz and Ozkut [4] optimized these objective functions for a parallel system consisting of dependent components.

This paper is concerned with a parallel system that has a random number of units. The lifetime of their units also following a copula since they are dependent. The distribution of the number of units is assumed to follow a Poisson distribution. The optimal replacement time for such a parallel system which minimizes the mean cost rate is computed. A parallel system with a random number of units is potentially useful in various real life situations. Consider a production system that consists of parallel machines (units). Assume that each day a certain number of items are produced by this system depending on the number of available units which may vary day by day. The variation in the number of units is caused by some factors such as malfunctioning of a machine and the insufficient number of operators. Thus, in a long term, we have a parallel system with a random number of units on hand. The statistical distribution of the number of units N can be determined by using daily data of the number of available units.

The optimal replacement time for a parallel system consists of some random number of dependent components discussed in Section 2. Section 3 provide corresponding numerical results for such a system and some of their reliability characteristics. Finally, the conclusion of our study is present in Section 4.

2 Main results

Consider a parallel system consisting of N components, such that the system fails when all components failed. Let $X_1, X_2, \ldots, X_N$ denote the failure times of the components when the parallel system consists of dependent and non-identical components. The number of components N, itself is random and distributed the same as Poisson distribution with non-negative parameters λ. The lifetime of all components is assumed following an absolutely continuous cumulative distribution function $P(X_i \leq x) = F_i(x)$. Moreover assuming copula function $C(.)$ as dependency model among the lifetime of components, we have:

$$P(X_1 \le x_1, X_2 \le x_2, \dots, X_N \le x_N | N = n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)), \quad (1)$$

where $C(.)$ satisfies in continuing relations.

I: $C(u_1, u_2, \dots, u_d) = 0$ if at least one $u_j = 0$.

II: $C(1, 1, \dots, 1, u_j, 1, \dots, 1, 1) = u_j$ if at most one $u_j \neq 1$.

III:

$$\int_B dC(u) = \sum_{\mathbf{z} \in \times_{i=1}^{d} \{x_i, y_i\}} (-1)^{N(\mathbf{z})} C(\mathbf{z}) \ge 0.$$

Then $X_{N:N}$ which is the largest among $X_1, X_2, \dots, X_N$ represents the lifetime of the parallel system. Therefore, the lifetime of the parallel system corresponds to $T = X_{N:N}$. Consequently using (1), the reliability function of such a parallel system is given by:

$$\begin{aligned}
S_T(t) &= P(T > t) \\
&= P(X_{N:N} > t) \\
&= 1 - P(X_{N:N} \le t) \\
&= 1 - P(X_1 \le t, X_2 \le t, \dots, X_N \le t) \\
&= 1 - \sum_{n=1}^{\infty} [P(X_1 \le t, X_2 \le t, \dots, X_N \le t | N = n) \times P(N = n)] \\
&= 1 - \sum_{n=1}^{\infty} [P(X_1 \le t, X_2 \le t, \dots, X_N \le t) \times P(N = n)] \\
&= 1 - \sum_{n=1}^{\infty} [C(F_1(t), F_2(t), \dots, F_n(t)) \times \frac{e^{-\lambda} \lambda^n}{n! (1 - e^{-\lambda})}].
\end{aligned} \quad (2)$$

Assuming $F_T(t) = 1 - S_T(t)$ the mean time to failure and mean cost rate function of the parallel system are given by:

$$\begin{aligned}
\mu &= E(X_{N:N}) \\
&= E(E(X_{N:N} | N))
\end{aligned}$$

$$= E(\int_0^\infty (1 - C(F_1(t), F_2(t), \ldots, F_n(t)))dt)$$

$$= \sum_{k=1}^{\infty} (\int_0^\infty (1 - C(F_1(t), F_2(t), \ldots, F_n(t)))dt) \frac{e^{-\lambda}\lambda^n}{n!\,(1-e^{-\lambda})}, \quad (3)$$

and

$$\begin{aligned} M(A) &= \frac{c_1 E(N) + c_2 F_T(A)}{E(min(T,A))} \\ &= c_1 \frac{E(N) + \frac{c_2}{c_1} F_T(A)}{E(min(T,A))} \\ &= c_1 \frac{\lambda + \frac{c_2}{c_1} F_T(A)}{\int_0^A S_T(t)dt}. \end{aligned} \quad (4)$$

In fact, the system is replaced at time A or at failure, whichever occurs first. A cost nc_1 is suffered for a non-failed system that is replaced at time A and a cost $nc_1 + c_2$ is suffered for a failed system. Obviously minimizing (4) heavily depends on the selection of the copula function $C(.)$ and parmeter λ. In addition if we assume that $\frac{c_2}{c_1} = cc$, minimizing (4) is equivalent to minimize

$$M(A) \propto \frac{\lambda + ccF_T(A)}{\int_0^A S_T(t)dt}. \quad (5)$$

Since $c_1 > 0$, and in continue we provide the optimal value of (5) instead of (4).

3 Simulation study

In this section we provide the valus of (3) and also minimize (5) with the following assumptions.

I: $cc = 5, 25.$

II: $\lambda = 1, 2, 3.$

III: For modeling dependency among the lifetime of components, the positive quadratic dependent is suitable [4]. Consequently, the Gumbel copula has been chosen, since its positive quadratic dependency properties and including independence copula. The Gumbel copula has the form

$$C(u_1,u_2,\ldots,u_n) = \exp[-(\sum_{k=1}^{n}(-\log(u_k))^{\theta})^{\frac{1}{\theta}}], \theta \geq 1. \quad (6)$$

The corresponding parameters of this copula are respectively considered $1,5,20$. The Gumbel copula with parameter 1 is an independent copula and two other parameters considered respectively as a weak and strong positive quadratic dependent.

IV: For the lifetime of components, we consider three cases including the Weibull and exponential distributions for heterogeneous form and homogeneous standard exponential distribution such that:

$$\begin{cases} I: & F_i(t) = 1 - e^{-it^i}, \\ II: & F_i(t) = 1 - e^{-it}, \\ III: & F_i(t) = 1 - e^{-t}. \end{cases}$$

For these given values, the mean time to failure, optimal replacement time, and its mean cost rate function are provided in Tables (1),(2), and (3). Regarding these tables, the following propositions can be mentioned.

Proposition 3.1. *The R code of calculating mean time to failure:*

```
rm(list=ls())
main=function(lam,theta){
require(copula);require(rmutil)
p=function(n)
{(exp(-lam)*(lam^n))/(factorial(n)*(1-exp(-lam)))}
p=Vectorize(p)
g=function(n){
if(n==1){
```

```
return(1)
}else{
co=mvdc(gumbelCopula(theta,dim=n),
rep("exp",n),rep(list((list(rate=1))),n)
surv=function(t)1-pMvdc(rep(t,n),co)
return(int(surv,0,Inf))}}
g=Vectorize(g)
mu=function(N)sum(g(1:N)*p(1:N))
return(mu(30))
}
```

Proposition 3.2. *The R code for optimization problems is also presented by:*

```
rm(list=ls())
require(DEoptim)
cc=5;lam=1;theta=5
G=function(tt){
require(copula);require(rmutil)
p=function(n)
{(exp(-lam)*(lam^n))/(factorial(n)*(1-exp(-lam)))}
p=Vectorize(p)
g=function(n,t){
if(n==1){
return(p(1)*(1-exp(-t)))
}
else{
co=mvdc(gumbelCopula(theta,dim=n),
rep("exp",n),rep(list((list(rate=1))),n))
surv=function(t)pMvdc(rep(t,n),co)
return(p(n)*surv(t))}}
g=Vectorize(g)
```

```
return(1-sum(g(1:25,tt)))}
object=function(aa)(lam+cc*(1-G(aa)))/(int(G,0,aa))
DEoptim(object,0,15)
```

4 Conclusion

A parallel system consisting of heterogeneous and dependent components has been considered. The dependency among their units is modeled utilizing a copula function. The lifetime of these components is assumed to following an absolutely continuous cumulative density function and the number of components is also randomly assumed following the Poisson probability mass function. For such a system, some reliability characteristics including the mean time to failure, reliability function, and mean cost rate function are calculated. The mean cost rate function is also optimized, giving the optimal replacement time of this system. Under different cases, such as different copula dependency, different Poisson parameters, and different lifetime assumptions, the derived characteristics are compared. Finally to the best of our knowledge about any different situations and assumptions, extensive simulation studies are presented through many tables and figures.

			$\lambda = 1$		
		$cc = 5$		$cc = 25$	
Case	μ	A_{opt}	$M(A_{opt})$	A_{opt}	$M(A_{opt})$
I	1.260	2.264	4.744	0.582	18.666
II	1.467	2.841	5.214	0.689	19.963
III	1.325	2.589	4.936	0.623	19.105
			$\lambda = 2$		
I	1.525	2.396	4.558	0.679	14.430
II	1.639	2.895	5.235	0.785	14.962
III	1.575	2.652	4.915	0.718	14.621
			$\lambda = 3$		
I	1.777	2.608	4.467	0.795	11.786
II	1.965	3.125	5.168	0.956	13.247
III	1.852	2.941	4.925	0.815	12.546

Table 1: The mean time to failure, optimal replacement time and its mean cost rate under given values, and Gumbel dependency with parameter 1, (independent copula).

			$\lambda = 1$		
		$cc = 5$		$cc = 25$	
Case	μ	A_{opt}	$M(A_{opt})$	A_{opt}	$M(A_{opt})$
I	1.046	2.036	4.128	0.496	18.003
II	1.132	2.136	4.221	0.519	18.268
III	1.147	2.365	4.269	0.524	18.496
			$\lambda = 2$		
I	1.093	1.963	4.189	0.258	14.089
II	1.295	2.458	4.863	0.378	14.587
III	1.174	2.284	4.521	0.397	14.389
			$\lambda = 3$		
I	1.135	2.149	3.998	0.269	11.248
II	1.623	2.759	4.675	0.568	12.874
III	1.358	2.628	4.519	0.328	12.009

Table 2: The mean time to failure, optimal replacement time and its mean cost rate under given values, and Gumbel dependency with parameter 5.

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			$\lambda = 1$		
		$cc = 5$		$cc = 25$	
Case	μ	A_{opt}	$M(A_{opt})$	A_{opt}	$M(A_{opt})$
I	1.011	1.963	4.002	0.369	17.562
II	1.075	2.006	3.956	0.489	17.951
III	1.112	2.296	4.203	0.482	18.375
			$\lambda = 2$		
I	1.022	1.815	3.896	0.254	17.149
II	1.065	1.985	3.741	0.471	17.748
III	1.096	2.185	4.178	0.476	18.194
			$\lambda = 3$		
I	1.032	1.874	3.935	0.317	17.387
II	1.079	2.011	3.895	0.492	17.962
III	1.122	2.281	4.236	0.514	18.629

Table 3: The mean time to failure, optimal replacement time and its mean cost rate under given values, and Gumbel dependency with parameter 20.

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On the Matrix Variate SSMESN Family of Distrbutions

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Abstract: Recently the scale and shape mixtures of matrix variate extended skew normal distributions (SSMESN) is introduced as a family of the matrix variate distributions. Problem of finding a Bayes estimation for the mean matrix of these distributions is considered and some applications are described for the result. Finally, a simulation study is presented for an application.

Keywords: Matrix variate SSMESN family, Multivariate linear regression model, Posterior density, Stress-strength reliability.

1 Introduction

In multivariate analysis methods, the matrix variate distributions are very important and have a key role. For example, the distribution of the maximum likelihood estimator of the covariance matrix of a multivariate normal distribution is the Wishart distribution which plays a pivotal role in related analysis. The matrix variate normal distribution is another important

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matrix variate distribution, see [1] and [3]. An $p \times n$ random matrix X is said to follow a matrix variate normal distribution if its probability density function (pdf) can be written as

$$\phi_{p\times n}(X;M,\Psi\otimes\Sigma) = (2\pi)^{-\frac{np}{2}}|\Psi|^{-\frac{p}{2}}|\Sigma|^{-\frac{n}{2}}etr\Big\{-\frac{1}{2}\Psi^{-1}(X-M)'\Sigma^{-1}(X-M)\Big\},$$

where M is an $p \times n$ mean matrix, Σ is an $p \times p$ positive definite matrix and Ψ is an $n \times n$ positive definite matrix. The normal matrix variate X is denoted by $X \sim N_{p\times n}(M,\Psi\otimes\Sigma)$. The matrix variate extended skew normal distribution, introduced by [5], is one of skew versions of the matrix variate normal distribution. An $p \times n$ random matrix X follows a matrix variate extended skew normal distribution with an $p \times n$ mean matrix M, an $p \times p$ positive definite matrix Σ and $n \times n$ positive definite matrices Ω and Ψ, if its pdf can be written as

$$f_{ESN}(X;M,\Psi\otimes\Sigma,\Omega,\lambda,\delta) = \frac{\phi_{p\times n}(X;M,\Psi\otimes\Sigma)}{\Phi_n(\delta;\Omega+\lambda'\lambda\Psi)}\Phi_n(\delta+(X-M)'\Sigma^{-\frac{1}{2}}\lambda;\Omega),$$

where λ and δ are p and q dimensional vectors, respectively, $\phi_{p\times n}(\cdot;M,\Psi\otimes\Sigma)$ is the pdf of $N_{p\times n}(M,\Psi\otimes\Sigma)$ and $\Phi_n(\cdot;\Omega)$ is the cumulative distribution function (cdf) of $N_n(0,\Omega)$. The extended skew normal matrix variate X is denoted by $X \sim ESN_{p\times n}(M,\Psi\otimes\Sigma,\Omega,\lambda,\delta)$.

[7] recently introduced the scale and shape mixtures of matrix variate extended skew normal distributions (SSMESN) as a new family of the distributions which includes a wide range of the matrix variate distributions such as normal, skew normal, t, skew t, skew-t-normal, skew-normal-Cauchy and etc. An $p \times n$ random matrix Y is said to follow a SSMESN distribution with an $p \times n$ mean matrix M, an $p \times p$ positive definite matrix Σ and $n \times n$ positive definite matrices Ω and Ψ, if

$$Y \mid \theta=\theta_0, \omega=\omega_0 \sim ESN_{p\times n}(M,\Psi\otimes k(\theta_0)\Sigma,\Omega,s(\theta_0,\omega_0)\lambda,\delta), \qquad (1)$$

where θ and ω are two random variables with joint distribution $Q(\theta_0,\omega_0)$ and marginal distributions $H(\theta_0)$ and $G(\omega_0)$, respectively, $k(\theta_0)$ is a weight function and $s(\theta_0,\omega_0)$ is a real valued function. The SSMESN matrix variate Y is denoted by $Y \sim SSMESN_{p\times n}(M,\Psi\otimes\Sigma,\Omega,\lambda,\delta;(k,s),Q)$. From (1), it is obvious that the pdf of Y is given by

$$f(Y;M,\Sigma,\Psi,\Omega,\lambda,\delta) = \int_{S_Q} f_{ESN}(Y;M,\Psi\otimes k(\theta)\Sigma,\Omega,s(\theta,\omega)\lambda,\delta)dQ(\theta,\omega), \quad (2)$$

where S_Q is the support of Q.

There is an important situation for the SSMESN matrix variate Y with the columns $y_1,\dots,y_n$ which is obtained by considering $M = 1_n'\otimes\mu$, $\delta = \delta 1_n$ and $\Omega = \Psi = I_n$, where $\mu \in \mathbb{R}^p$, $\delta \in \mathbb{R}^1$ and 1_n is a n-dimensional vector of ones. In this situation,

$$y_i \mid (\theta,\omega) \overset{iid}{\sim} ESN_p(\mu,k(\theta)\Sigma,s(\theta,\omega)\lambda,\delta), \quad i = 1,\dots,n,$$

with the conditional pdf

$$\begin{aligned} f_{ESN}(y_i \mid \theta,\omega;\mu,k(\theta)\Sigma,s(\theta,\omega)\lambda,\delta) &= \frac{1}{\Phi_1(\delta/\sqrt{1+s(\theta,\omega)^2\lambda'\lambda})}\phi_p(y_i;\mu,k(\theta)\Sigma) \\ &\times \Phi_1(\delta + s(\theta,\omega)k(\theta)^{-\frac{1}{2}}(y_i-\mu)'\Sigma^{-\frac{1}{2}}\lambda), y_i \in \mathbb{R}^p, \end{aligned}$$

where ϕ_p and Φ_1 are the pdf of the p-variate normal distribution and the cdf of the univariate standard normal distribution, respectively.

The matrix variate SSMESN family is quite large and includes some different matrix variate distributions. For example,

- If $k(\theta_0) = s(\theta_0,\omega_0) = 1$, we have the matrix variate extended skew normal distribution.
- If $k(\theta_0) = s(\theta_0,\omega_0) = 1$ and $\lambda = 0$, then the matrix variate normal distribution is obtained.
- If $\lambda = 0$, then we obtain the scale mixture of matrix variate normal distributions which proposed by [2]. We denote this subfamily by $SMN_{p\times n}(M,\Psi\otimes\Sigma;k,H)$.
- If $\delta = 0$, then the SSMESN matrix variate Y follows the matrix variate skew t distribution with ν degrees of freedom by considering $k(\theta_0) =$

θ_0 and $s(\theta_0, \omega_0) = 1$ with $\theta \sim IGamma(\frac{\nu}{2}, \frac{\nu}{2})$, where $IGamma(a,b)$ denotes the inverse gamma distribution with shape parameter a and scale parameter b. We use the notation $ST_{p\times n}(M, \Psi \otimes \Sigma, \Omega, \lambda, \nu)$ to denote this distribution.

- If $\delta = 0$, $\Psi = \Omega = I_n$, $k(\theta_0) = 1$ and $s(\theta_0, \omega_0) = \omega_0^{-\frac{1}{2}}$ with $\omega \sim IGamma(\frac{1}{2}, \frac{1}{2})$, then the SSMESN matrix variate Y follows the matrix variate skew-normal-Cauchy distribution which is denoted here by $SNC_{p\times n}(M, \Sigma, \lambda)$.

In next section, we obtain a posterior density for the mean matrix of the matrix variate SSMESN distributions. Also, applications of the obtained result in multivariate linear regression and stress-strength models will be discussed in Section 3. Finally, Section 4 will present a simulation study for comparing the Bayes estimators of a stress-strength reliability.

2 Main Result

To find a Bayes estimation for a parameter, here the mean matrix, it must minimize the posterior risk. For this, the posterior distribution or posterior density should be used. In this section, a posterior density for the mean matrix of the matrix variate SSMESN distributions is derived by considering a matrix variate normal distribution as prior. The result is given in the following proposition.

Proposition 2.1. *Suppose that $Y \sim SSMESN_{p\times n}(M, \Psi \otimes \Sigma, \Omega, \lambda, \delta; (k,s), Q)$ where Σ, Ψ, Ω, λ and δ are known. Let M is independent of θ and ω, and has prior distribution as $N_{p\times n}(0_{p\times n}, \Psi \otimes \Delta)$, where $\Delta_{p\times p}$ is a positive definite matrix. Then the posterior density of M is*

$$\frac{\int_{S_Q} \frac{\rho_\theta |\Lambda_\theta|^{\frac{n}{2}} \phi_{p\times n}(M; \Lambda_\theta \tau \Psi, \Psi \otimes k(\theta)\Lambda_\theta) \Phi_n(\delta + s(\theta,\omega) k(\theta)^{-\frac{1}{2}} (Y-M)' \Sigma^{-\frac{1}{2}} \lambda; \Omega)}{\Phi_n(\delta; \Omega + s(\theta,\omega)^2 \lambda' \lambda \Psi)} dQ(\theta,\omega)}{\int_{S_Q} \frac{\rho_\theta |\Lambda_\theta|^{\frac{n}{2}} \Phi_n(\delta + s(\theta,\omega) k(\theta)^{-\frac{1}{2}} (Y - \Lambda_\theta \tau \Psi)' \Sigma^{-\frac{1}{2}} \lambda; \Omega + s(\theta,\omega)^2 \lambda' \Sigma^{-\frac{1}{2}} \Lambda_\theta \Sigma^{-\frac{1}{2}} \lambda \Psi)}{\Phi_n(\delta; \Omega + s(\theta,\omega)^2 \lambda' \lambda \Psi)} dQ(\theta,\omega)}, \quad (3)$$

where $\Lambda_\theta = (\Sigma^{-1} + k(\theta)\Delta^{-1})^{-1}$, $\tau = \Sigma^{-1}Y\Psi^{-1}$ *and* $\rho_\theta = etr\left\{\frac{\Lambda_\theta\tau\Psi\tau' - \tau Y'}{2k(\theta)}\right\}$.

Proof. From the pdfs of Y and M, we have

$$\begin{aligned} f(Y|M)\pi(M) \;\propto\; & \int_{S_Q} \frac{(2\pi)^{-\frac{np}{2}}|\Psi|^{-\frac{p}{2}}k(\theta)^{-\frac{np}{2}}}{\Phi_n(\delta;\Omega+s(\theta,\omega)^2\lambda'\lambda\Psi)} \\ & \times etr\left\{\frac{-1}{2k(\theta)}\Psi^{-1}(M-Y)'\Sigma^{-1}(M-Y) - \frac{1}{2}\Psi^{-1}M'\Delta^{-1}M\right\} \\ & \times \Phi_n(\delta + s(\theta,\omega)k(\theta)^{-\frac{1}{2}}(Y-M)'\Sigma^{-\frac{1}{2}}\lambda;\Omega)dQ(\theta,\omega). \end{aligned}$$

Since,

$$etr\left\{\frac{-1}{2k(\theta)}\Psi^{-1}(M-Y)'\Sigma^{-1}(M-Y)\right\} = etr\left\{\frac{-1}{2k(\theta)}\tau Y'\right\}etr\left\{-\frac{1}{2k(\theta)}\Psi^{-1}M'\Sigma^{-1}M + \frac{1}{k(\theta)}\tau' M\right\}$$

and

$$\begin{aligned} etr\left\{\frac{-1}{2k(\theta)}[\Psi^{-1}M'\Lambda_\theta^{-1}M - 2\tau' M]\right\} \;=\; & etr\left\{\frac{1}{2k(\theta)}\Lambda_\theta\tau\Psi\tau'\right\} \\ & \times etr\left\{\frac{-1}{2k(\theta)}\Psi^{-1}(M-\Lambda_\theta\tau'\Psi)'\Lambda_\theta^{-1}(M-\Lambda_\theta\tau'\Psi)\right\}, \end{aligned}$$

we can write

$$\begin{aligned} f(Y|M)\pi(M) \;\propto\; & \int_{S_Q} etr\left\{\frac{\Lambda_\theta\tau\Psi\tau' - \tau Y'}{2k(\theta)}\right\}|\Lambda_\theta|^{\frac{n}{2}}\frac{\phi_{p\times n}(M;\Lambda_\theta\tau\Psi,\Psi\otimes k(\theta)\Lambda_\theta)}{\Phi_n(\delta;\Omega+s(\theta,\omega)^2\lambda'\lambda\Psi)} \\ & \times \Phi_n(\delta + s(\theta,\omega)k(\theta)^{-\frac{1}{2}}(Y-M)'\Sigma^{-\frac{1}{2}}\lambda;\Omega)dQ(\theta,\omega). \end{aligned}$$

So, by substituting $U = M - \Lambda_\theta\tau\Psi$,

$$\begin{aligned} \int_{\mathbb{R}^{p\times n}} f(Y|M)\pi(M)dM \;\propto\; & \int_{S_Q} \frac{etr\left\{\frac{\Lambda_\theta\tau\Psi\tau' - \tau Y'}{2k(\theta)}\right\}|\Lambda_\theta|^{\frac{n}{2}}}{\Phi_n(\delta;\Omega+s(\theta,\omega)^2\lambda'\lambda\Psi)}E[\Phi_n(\delta + s(\theta,\omega)k(\theta)^{-\frac{1}{2}} \\ & \times (Y-\Lambda_\theta\tau\Psi)'\Sigma^{-\frac{1}{2}}\lambda - s(\theta,\omega)k(\theta)^{-\frac{1}{2}}U'\Sigma^{-\frac{1}{2}}\lambda;\Omega)]dQ(\theta,\omega), \end{aligned}$$

where $U \sim N_{p\times n}(0_{p\times n},\Psi\otimes k(\theta)\Lambda_\theta)$.

We know by Lemma 2.1 of [4],

$$\begin{aligned} & E[\Phi_n(\delta + s(\theta,\omega)k(\theta)^{-\frac{1}{2}}(\mathscr{Y}-\Lambda_\theta\tau\Psi)'\Sigma^{-\frac{1}{2}}\lambda - s(\theta,\omega)k(\theta)^{-\frac{1}{2}}U'\Sigma^{-\frac{1}{2}}\lambda;\Omega)] \\ & \quad = \Phi_n(\delta + s(\theta,\omega)k(\theta)^{-\frac{1}{2}}(\mathscr{Y}-\Lambda_\theta\tau\Psi)'\Sigma^{-\frac{1}{2}}\lambda;\Omega + s(\theta,\omega)^2\lambda'\Sigma^{-\frac{1}{2}}\Lambda_\theta\Sigma^{-\frac{1}{2}}\lambda\Psi), \end{aligned}$$

and the proof is complete by

$$\pi(M|Y) = \frac{f(Y|M)\pi(M)}{\int_{\mathbb{R}^{p\times n}} f(Y|M)\pi(M)dM}.$$

□

The following corollaries can be written by using Proposition 2.1.

Corollary 2.2. *Let* $\Lambda = (\Sigma^{-1} + \Delta^{-1})^{-1}$.

(i) If $Y \sim N_{p\times n}(M, \Psi \otimes \Sigma)$, *then* $M|Y \sim N_{p\times n}(\Lambda\Sigma^{-1}Y, \Psi \otimes \Lambda)$.

(ii) If $Y \sim ESN_{p\times n}(M, \Psi \otimes \Sigma, \Omega, \lambda, \delta)$, *then the posterior distribution of M is*

$$ESN_{p\times n}(\Lambda\Sigma^{-1}Y, \Psi \otimes \Lambda, \Omega, -\Lambda^{\frac{1}{2}}\Sigma^{-\frac{1}{2}}\lambda, \delta + (Y - \Lambda\Sigma^{-1}Y)'\Sigma^{-\frac{1}{2}}\lambda).$$

Corollary 2.3. *If* $Y \sim ST_{p\times n}(M, \Psi \otimes \Sigma, \Omega, \lambda, \nu)$, *then the posterior density of M is as follows:*

$$\frac{E_\theta\left[\rho_\theta|\Lambda_\theta|^{\frac{n}{2}}\phi_{p\times n}(M; \Lambda_\theta\tau\Psi, \Psi \otimes \theta\Lambda_\theta)\Phi_n(\theta^{-\frac{1}{2}}(Y - M)'\Sigma^{-\frac{1}{2}}\lambda; \Omega)\right]}{E_\theta\left[\rho_\theta|\Lambda_\theta|^{\frac{n}{2}}\Phi_n(\theta^{-\frac{1}{2}}(Y - \Lambda_\theta\tau\Psi)'\Sigma^{-\frac{1}{2}}\lambda; \Omega + \lambda'\Sigma^{-\frac{1}{2}}\Lambda_\theta\Sigma^{-\frac{1}{2}}\lambda\Psi)\right]},$$

where $\tau = \Sigma^{-1}Y\Psi^{-1}$, $\Lambda_\theta = (\Sigma^{-1} + \theta\Delta^{-1})^{-1}$ *and* $\rho_\theta = etr\left\{\frac{\Lambda_\theta\tau\Psi\tau' - \tau Y'}{2\theta}\right\}$.

Corollary 2.4. *If* $Y \sim SNC_{p\times n}(M, \Sigma, \lambda)$, *then*

$$\pi(M|Y) = \frac{\phi_{p\times n}(M; \Lambda\tau, I_n \otimes \Lambda)F_C((Y - M)'\Sigma^{-\frac{1}{2}}\lambda; I_n)}{E_\omega\left[\Phi_n\left(\frac{(Y - \Lambda\tau)'\Sigma^{-\frac{1}{2}}\lambda}{\sqrt{\omega + \lambda'\Sigma^{-\frac{1}{2}}\Lambda\Sigma^{-\frac{1}{2}}\lambda}}; I_n\right)\right]},$$

where $\tau = \Sigma^{-1}Y$, $\Lambda = (\Sigma^{-1} + \Delta^{-1})^{-1}$ *and* $F_C(\cdot; I_n)$ *is the cdf of the n-variate standard Cauchy distribution.*

3 Applications

The result obtained in Proposition 2.1 can be used in many models. In this section, we explain the application of the result in the multivariate linear regression and stress-strength models.

3.1 Multivariate linear regression models

The following corollary derives a posterior density for the parameters in the multivariate linear regression models.

Corollary 3.1. *Suppose that $x_1,\dots,x_n$ are p-dimensional vectors such that*

$$x_i \overset{iid}{\sim} SMN_p(Bz_i,\Sigma;k,H), \quad i=1,\dots,n, \tag{4}$$

where z_i is a q-dimensional known vector and B is a $p\times q$ unknown matrix. If B has prior distribution as $N_{p\times q}(0_{p\times q},(ZZ')^{-1}\otimes\Xi)$, where $Z=(z_1,\dots,z_n)$ is a $q\times n$ known matrix and $\Xi_{p\times p}$ is a positive definite matrix. Then the posterior density of B, the regression parameters, is given by

$$\Big[\int_{S_H} etr\{\frac{1}{2k(\theta)}(\Pi_\theta\Sigma^{-1}-I_p)XZ'(ZZ')^{-1}ZX'\Sigma^{-1}\}|\Pi_\theta|^{\frac{q}{2}}dH(\theta)\Big]^{-1}$$
$$\times\int_{S_H} etr\{\frac{1}{2k(\theta)}(\Pi_\theta\Sigma^{-1}-I_p)XZ'(ZZ')^{-1}ZX'\Sigma^{-1}\}|\Pi_\theta|^{\frac{q}{2}}$$
$$\times\phi_{p\times q}(B;\Pi_\theta\Sigma^{-1}XZ'(ZZ')^{-1},(ZZ')^{-1}\otimes k(\theta)\Pi_\theta)dH(\theta),$$

where $X=(x_1,\dots,x_n)_{p\times n}$ and $\Pi_\theta=(\Sigma^{-1}+k(\theta)\Xi^{-1})^{-1}$.

Proof. From (4), it follows that

$$X\sim SMN_{p\times n}(BZ,I_n\otimes\Sigma;k,H).$$

Since $X\mid\theta\sim N_{p\times n}(BZ,I_n\otimes k(\theta)\Sigma)$, by properties of the matrix variate normal distribution, we have

$$XZ'(ZZ')^{-1}\sim SMN_{p\times q}(B,(ZZ')^{-1}\otimes\Sigma;k,H).$$

Now, since $B\sim N_{p\times q}(0_{p\times q},(ZZ')^{-1}\otimes\Xi)$. The proof is completed by using Proposition 2.1 with $Y_{p\times q}=XZ'(ZZ')^{-1}$, $\Psi_{q\times q}=(ZZ')^{-1}$, $\Delta=\Xi$ and $\lambda=0$. □

3.2 Stress-strength models

Another application of the result of Proposition 2.1 is obtaining a Bayes estimator for the reliability of the stress-strength model. We describe it as follows;

In the stress-strength model, the reliability is $R = P(a'x + b'y + c > 0)$, where $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are two independent random vectors and $a \in \mathbb{R}^p$, $b \in \mathbb{R}^q$ and $c \in \mathbb{R}$ are known. Here, suppose that

$$x \mid (\theta, \omega) \sim ESN_p(\mu_1, k(\theta)\Sigma_1, s(\theta, \omega)\lambda_1, \delta_1), \tag{5}$$

and

$$y \mid (\theta, \omega) \sim ESN_q(\mu_2, k(\theta)\Sigma_2, s(\theta, \omega)\lambda_2, \delta_2), \tag{6}$$

and denote the corresponding reliability by R_{SSMESN}.

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are two independent random samples from the distributions x and y, respectively. It is obvious that,

$$X = (x_1, \ldots, x_n) \sim SSMESN_{p\times n}(M_1, I_n \otimes \Sigma_1, I_n, \lambda_1, \delta_1; (k, s), Q),$$

and

$$Y = (y_1, \ldots, y_m) \sim SSMESN_{q\times m}(M_2, I_m \otimes \Sigma_2, I_m, \lambda_2, \delta_2; (k, s), Q),$$

where $M_1 = 1_n' \otimes \mu_1$, $\delta_1 = \delta_1 1_n$, $M_2 = 1_m' \otimes \mu_2$ and $\delta_2 = \delta_2 1_m$. Let Σ_i, λ_i and δ_i, for $i = 1, 2$, are known and consider prior distributions $N_{p\times n}(0_{p\times n}, I_n \otimes \Delta_1)$ and $N_{q\times m}(0_{q\times m}, I_m \otimes \Delta_2)$ for M_1 and M_2, respectively. By Proposition 2.1, posterior densities of M_1 and M_2 have the form (3). Since $\mu_1 = \frac{1}{n} M_1 1_n$ and $\mu_2 = \frac{1}{m} M_2 1_m$, R_{SSMESN} is a function of M_1 and M_2, i.e. $R_{SSMESN}(M_1, M_2)$, and its Bayes estimator under squared error loss function is obtained by

$$\hat{R}_{SSMESN} = \int_{\mathbb{R}^{p\times n}} \int_{\mathbb{R}^{q\times m}} R_{SSMESN}(M_1, M_2)\pi(M_1|X)\pi(M_2|Y)d\mathcal{M}_2 d\mathcal{M}_1. \tag{7}$$

In the following, we present examples for obtaining the stress-strength reliability of some multivariate distributions such as normal, skew t and skew-normal-Cauchy.

Example 3.2. By considering $k(\theta) = s(\theta,\omega) = 1$, $\lambda_1 = 0$, $\lambda_2 = 0$ and $\delta_1 = \delta_2 = 0$, the Bayes estimator for the stress-strength reliability corresponding to the multivariate normal distributions, R_N, is obtained from (7), where

$$R_{SSMESN}(M_1,M_2) \equiv R_N(M_1,M_2) = \Phi_1\left(\frac{\frac{1}{n}a'M_1 1_n + \frac{1}{m}b'M_2 1_m + c}{\sqrt{a'\Sigma_1 a + b'\Sigma_2 b}}\right),$$

and by Corollary 2.2,

$$M_1|X \sim N_{p\times n}(\Lambda_1\Sigma_1^{-1}X, I_n\otimes\Lambda_1), \quad \text{and} \quad M_2|Y \sim N_{q\times m}(\Lambda_2\Sigma_2^{-1}Y, I_m\otimes\Lambda_2)$$

with $\Lambda_i = (\Sigma_i^{-1} + \Delta_i^{-1})^{-1}$ for $i = 1,2$.

Example 3.3. If it is considered in (5) and (6), respectively,

$$\delta_1 = 0, k(\theta_0) = \theta_0, s(\theta_0,\omega_0) = 1, \theta_1 \sim IGamma(\tfrac{\nu_1}{2},\tfrac{\nu_1}{2}),$$

and

$$\delta_2 = 0, k(\theta_0) = \theta_0, s(\theta_0,\omega_0) = 1, \theta_2 \sim IGamma(\tfrac{\nu_2}{2},\tfrac{\nu_2}{2}),$$

then $x \sim ST_p(\mu_1,\Sigma_1,\lambda_1;\nu_1)$ and $y \sim ST_q(\mu_2,\Sigma_2,\lambda_2;\nu_2)$. In this case, the reliability R_{SSMESN} becomes the stress-strength reliability of the multivariate skew t distributions, R_{ST}, which has been calculated by [6]. Hence, by using (7), the Bayes estimator of R_{ST} is as follows:

$$\hat{R}_{ST}^{Bayes} = \int_{\mathbb{R}^{p\times n}}\int_{\mathbb{R}^{q\times m}} R_{ST}(M_1,M_2)\pi(M_1|X)\pi(M_2|Y)dM_2dM_1,$$

where from Corollary 2.3,

$$\pi(M_1|X) = \frac{E_{\theta_1}\left[\rho_{\theta_1}^1|\Lambda_{\theta_1}^1|^{\frac{n}{2}}\phi_{p\times n}(M_1;\Lambda_{\theta_1}^1\tau_1,I_n\otimes\theta_1\Lambda_{\theta_1}^1)\Phi_n(\theta_1^{-\frac{1}{2}}(X-M_1)'\Sigma_1^{-\frac{1}{2}}\lambda_1;I_n)\right]}{E_{\theta_1}\left[\rho_{\theta_1}^1|\Lambda_{\theta_1}^1|^{\frac{n}{2}}\Phi_n(\theta^{-\frac{1}{2}}(X-\Lambda_{\theta_1}^1\tau_1)'\Sigma_1^{-\frac{1}{2}}\lambda_1;(1+\lambda_1'\Sigma_1^{-\frac{1}{2}}\Lambda_{\theta_1}^1\Sigma_1^{-\frac{1}{2}}\lambda_1)I_n)\right]},$$

and

$$\pi(M_2|Y) = \frac{E_{\theta_2}\left[\rho_{\theta_2}^2|\Lambda_{\theta_2}^2|^{\frac{m}{2}}\phi_{q\times m}(M_2;\Lambda_{\theta_2}^2\tau_2,I_m\otimes\theta_2\Lambda_{\theta_2}^2)\Phi_{n_2}(\theta_2^{-\frac{1}{2}}(Y-M_2)'\Sigma_2^{-\frac{1}{2}}\lambda_2;I_m)\right]}{E_{\theta_2}\left[\rho_{\theta_2}^2|\Lambda_{\theta_2}^2|^{\frac{m}{2}}\Phi_m(\theta_2^{-\frac{1}{2}}(Y-\Lambda_{\theta_2}^2\tau_2)'\Sigma_2^{-\frac{1}{2}}\lambda_2;(1+\lambda_2'\Sigma_2^{-\frac{1}{2}}\Lambda_{\theta_2}^2\Sigma_2^{-\frac{1}{2}}\lambda_2)I_m)\right]},$$

with $\Lambda_{\theta_i}^i = (\Sigma_i^{-1} + \theta_i\Delta_i^{-1})^{-1}$ for $i = 1,2$, $\tau_1 = \Sigma_1^{-1}X$, $\tau_2 = \Sigma_2^{-1}Y$, $\rho_{\theta_1}^1 = etr\left\{\frac{\Lambda_{\theta_1}^1\tau_1\tau_1' - \tau_1X'}{2\theta_1}\right\}$ and $\rho_{\theta_2}^2 = etr\left\{\frac{\Lambda_{\theta_2}^2\tau_2\tau_2' - \tau_2Y'}{2\theta_2}\right\}$.

Example 3.4. By considering $\delta_1 = \delta_2 = 0$, $k(\theta_0) = \theta_0$ and $s(\theta_0, \omega_0) = \omega_0^{-\frac{1}{2}}$ in (5) and (6) when $\omega \sim IGamma(\frac{1}{2}, \frac{1}{2})$, the distribution of the vectors x and y are $SNC_p(\mu_1, \Sigma_1, \lambda_1)$ and $SNC_q(\mu_2, \Sigma_2, \lambda_2)$, respectively. Consider the stress-strength reliability of these vectors and denote it by R_{SNC}. From (7), the Bayes estimator of R_{SNC} is given by

$$\hat{R}_{SNC}^{Bayes} = \int_{\mathbb{R}^{p\times n}} \int_{\mathbb{R}^{q\times m}} R_{SNC}(M_1, M_2)\pi(M_1|X)\pi(M_2|Y)dM_2dM_1,$$

where by Corollary 2.4,

$$\pi(M_1|X) = \frac{\phi_{p\times n}(M_1; \Lambda_1\tau_1, I_n \otimes \Lambda_1)F_C((X - M_1)'\Sigma_1^{-\frac{1}{2}}\lambda_1; I_n)}{E_\omega\left[\Phi_n\left(\frac{(X-\Lambda_1\tau_1)'\Sigma_1^{-\frac{1}{2}}\lambda_1}{(\omega+\lambda_1'\Sigma_1^{-\frac{1}{2}}\Lambda_1\Sigma_1^{-\frac{1}{2}}\lambda_1)^{\frac{1}{2}}}; I_n\right)\right]},$$

$$\pi(M_2|Y) = \frac{\phi_{q\times m}(M_2; \Lambda_2\tau_2, I_m \otimes \Lambda_2)F_C((Y - M_2)'\Sigma_2^{-\frac{1}{2}}\lambda_2; I_m)}{E_\omega\left[\Phi_m\left(\frac{(Y-\Lambda_2\tau_2)'\Sigma_2^{-\frac{1}{2}}\lambda_2}{(\omega+\lambda_2'\Sigma_2^{-\frac{1}{2}}\Lambda_2\Sigma_2^{-\frac{1}{2}}\lambda_2)^{\frac{1}{2}}}; I_m\right)\right]},$$

with $\tau_1 = \Sigma_1^{-1}X$, $\tau_2 = \Sigma_2^{-1}Y$ and $\Lambda_i = (\Sigma_i^{-1} + \Delta_i^{-1})$ for $i = 1, 2$, and also by [6],

$$\begin{aligned} R_{SNC}(M_1, M_2) =& R_N(M_1, M_2) + \frac{1}{\pi}\Bigg[\int_0^\infty \frac{\cos\left(\left(\frac{1}{n}a'M_1 1_n + \frac{1}{m}b'M_2 1_m + c\right)u\right)}{u} \\ & \times e^{\frac{-u^2}{2}(a'\Sigma_1 a + b'\Sigma_2 b)}\left(\overset{*}{\tau}_{\Sigma_1,\lambda_1}(au) + \overset{*}{\tau}_{\Sigma_2,\lambda_2}(bu)\right)du \\ & - \int_0^\infty \frac{\sin\left(\left(\frac{1}{n}a'M_1 1_n + \frac{1}{m}b'M_2 1_m + c\right)u\right)}{u} \\ & \times e^{\frac{-u^2}{2}(a'\Sigma_1 a + b'\Sigma_2 b)}\ \overset{*}{\tau}_{\Sigma_1,\lambda_1}(au)\ \overset{*}{\tau}_{\Sigma_2,\lambda_2}(bu)du\Bigg], \end{aligned}$$

where $\overset{*}{\tau}_{\Sigma,\lambda}(t) = \int_0^\infty \tau\left(\frac{\lambda'\Sigma^{\frac{1}{2}}t}{\sqrt{1+x^2\lambda'\lambda}}x\right)\phi_1(x)dx$ with the pdf of the univariate standard normal distribution, ϕ_1, and $\tau(z) = \sqrt{\frac{2}{\pi}}\int_0^z \exp\{\frac{t^2}{2}\}dt$.

4 Simulation study

In this section, we present a simulation study to compare the Bayes estimators of the stress-strength reliability of the multivariate skew normal-Cauchy distributions, R_{SNC}, with different priors. Here, we focus on comparing the Bayes estimators of the stress-strength reliability corresponding to

$$x \sim SNC_3\left(\mu_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \Sigma_1 = \begin{pmatrix} 4.5 & 1.5 & -0.4 \\ 1.5 & 3.0 & 2.3 \\ -0.4 & 2.3 & 4.5 \end{pmatrix}, \lambda_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right),$$

and

$$y \sim SNC_3\left(\mu_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \Sigma_2 = \begin{pmatrix} 2.0 & -0.5 & -0.8 \\ -0.5 & 2.2 & -0.3 \\ -0.8 & -0.3 & 1.8 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0.3 \\ -0.3 \\ -0.3 \end{pmatrix}\right),$$

with $a = b = (0.25, 0.5, -0.25)'$ and $c = 1$ which equals to 0.82164. We have taken the following priors to estimate R_{SNC}:

- **Prior-1**: $\Delta_1 = I_3$ and $\Delta_2 = I_3$;
- **Prior-2**: $\Delta_1 = \begin{pmatrix} 2.73 & -0.66 & -1.59 \\ -0.66 & 2.73 & 1.35 \\ -1.59 & 1.35 & 2.73 \end{pmatrix}$, and $\Delta_2 = \begin{pmatrix} 1.66 & 1.66 & 1.42 \\ 1.66 & 3.55 & 1.66 \\ 1.42 & 1.66 & 2.81 \end{pmatrix}$;
- **Prior-3**: $\Delta_1 = \begin{pmatrix} 1.66 & 1.66 & 1.42 \\ 1.66 & 3.55 & 1.66 \\ 1.42 & 1.66 & 2.81 \end{pmatrix}$, and $\Delta_2 = \begin{pmatrix} 2.73 & -0.66 & -1.59 \\ -0.66 & 2.73 & 1.35 \\ -1.59 & 1.35 & 2.73 \end{pmatrix}$.

Table 1 presents Bias and MSE of different Bayes estimators of R_{SNC} for varying samples sizes. It can be observed that for any pair (n,m), MSE and absolute value of Bias of the Bayes estimator with Prior-3 are lower than that of other Bayes estimators. Hence, Prior-3 introduces considerably more prior information for R_{SNC}. Also, as the sample sizes increase, MSE of each priors decrease.

Table 1: The results of simulation for different Bayes estimates of R_{SNC}.

(n,m)	Prior-1			Prior-2			Prior-3		
	Estimate	Bias	MSE	Estimate	Bias	MSE	Estimate	Bias	MSE
$(10,10)$	0.745261	-0.076379	0.007716	0.758075	-0.063564	0.010236	0.844336	0.022696	0.001999
$(10,15)$	0.779812	-0.041828	0.004289	0.746165	-0.075475	0.010454	0.852224	0.030584	0.002575
$(10,20)$	0.768550	-0.053089	0.004330	0.756174	-0.065466	0.007348	0.839514	0.017873	0.001680
$(15,10)$	0.777213	-0.044426	0.004964	0.748355	-0.073285	0.008810	0.833099	0.011459	0.002159
$(15,15)$	0.772117	-0.049522	0.004271	0.764487	-0.057152	0.007266	0.855693	0.034053	0.003166
$(15,20)$	0.772982	-0.048657	0.003523	0.754297	-0.067342	0.007193	0.826814	0.005173	0.002467
$(20,10)$	0.782422	-0.039217	0.004371	0.777499	-0.044141	0.007374	0.846645	0.025004	0.001743
$(20,15)$	0.788783	-0.032857	0.002021	0.755248	-0.066391	0.007225	0.834473	0.013092	0.001603
$(20,20)$	0.779591	-0.042048	0.002895	0.759800	-0.061840	0.006779	0.823567	0.001927	0.001908

Conclusion

In this paper, we have presented a posterior density for the mean matrix of the matrix variate SSMESN distributions, by considering a matrix variate normal distribution as its prior, that is an important result for this type of distributions. Also, we used the obtained posterior density for estimating in the multivariate linear regression and stress-strength models. Finally, we compared different Bayes estimator of the stress-strength reliability of the multivariate skew normal-Cauchy distributions by a simulation study.

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Mean Residual Life of Degrading Complex Systems with Intact Component at Time t

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Abstract: This article investigates the mean residual life for a complex k-out-of-n:G system that consists of n element each having two dependent components under degradation performance. A flexible copula-based multivariate model is considered for describing the dependence structure within the components. Assuming degradation path of each component comes from the Inverse Gaussian process, the mean residual life of a complex k-out-of-n:G system with intact elements at time t is obtained based on Frank copula. Moreover, a simulation study is provided to discuss how does the dependence of components within each element affect the system mean residual lifetime.

Keywords: Complex system, Copula function, Degradation performance, Mean residual life.

1 Introduction

The study of mean residual life of a coherent system has gained a great attention in reliability theory. The mean residual life function of coherent systems is a very important concept in reliability theory and survival

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analysis. For a coherent system with lifetime T and components' lifetimes $T_1, T_2, ..., T_n$ the usual mean residual life is defined by $E(T-t|T>t)$. In addition to the classical definition of the mean residual life, different versions of mean residual life functions have been defined for a coherent system. For instance, Asadi and Bayramoglu [1] have defined the mean residual life of a coherent system as $E(T-t|T_{1:n}>t)$, where $T_{r:n}$ stand for the rth smallest lifetime among $T_1, T_2, ..., T_n$. See, also Navarro [10] and Navarro and Durante [9]. Eryilmaz et al. [5] studied the mean residual life of coherent systems consisting of multiple types of dependent components
Many modern products such as electrical and electromechanical products have extremely high reliability, So the failures may not occur during short time at normal conditions. In such situations, it is difficult to assess the reliability of these products with traditional life tests. Fortunately, most of the highly reliable products have characteristics whose their degradations over time can be measured. Thus, we can provide useful reliability information to assess the reliability of the modern products using degradation data.

The stochastic process model treats degradation measurements as the realization of a stochastic process, such as Inverse Gaussian (IG) process [16, 21], wiener process [20, 7] and Gamma process [3, 4]. Using an adaptive Wiener process model, Zhai and Ye [22] investigated the residual life prediction of deteriorating products. Nezakati and Razmkhah [13] and Nezakati et al. [12] investigated reliability of k-out-of-n systems under degradation performance. Most of previous studies has dealt with only one performance characteristic or component failure mechanism level. In recent years, due to the flexibility of copula function, the modeling of multiple degradation processes via copula function has received a lot of attention [6]. For instance, Peng et al. [17] proposed a bivariate modeling based on IG process via Gaussian copula and applied it on a degradation dataset from heavy machine tools. Pan et al. [15] applied Frank copula with Wiener process as marginal. Wang et al. [19] studied the residual life estimation based on bivariate non-stationary gamma degradation process via Frank copula. Palayangoda and Ng [14] developed semiparametric and nonparametric approaches to model bivariate degradation processes. All of these studies, consider a single-element system with multiple components. In practice, modern products usually have complex structure with many functions. Bairamov [2] has considered systems that consist of n elements, each containing two dependent components. In this work, we consider a degrading complex k-out-of-n:G system with two components in each element and assume that the degradation of each component over time is governed by an IG process. Moreover, we assume that the two components are dependent and their dependency being characterized by Frank copula. Under these assumptions, we study the MRL function of the complex k-out-of-n:G system. Moreover, the effect of dependence structure on system reliability is investigated. The rest of this paper is organized as follows. In Section 2, some preliminaries are presented containing the IG process and time-to-failure distribution. Section 3 elaborates the modeling framework of a degrading k-out-of-n:G system with dependent degradation processes. The mean residual life functions of these complex systems with intact components at time t is investigated in section 4. Some graphical analyses are provided in section 5 to demonstrate the sensitivity of the MRL function with respect to dependence structure. Finally, some the proposed model is described in details.

2 Preliminaries

In current work, a degradation process over time, is modeled through the IG process. The IG process $\{X(t), \quad t \geq 0\}$ is defined as the stochastic process satisfying:

a) $X(t)$ has independent increments.

b) $X(t)-X(s)$ follows an IG distribution $IG(\mu(\Lambda(t)-\Lambda(s)),\nu[\Lambda(t)-\Lambda(s)]^2)$ for all $t>s>0$,

where $\Lambda(t)$ is a monotone increasing function and $IG(a,b)$, $a,b>0$, denotes the IG distribution with mean a and variance a^3/b. The pdf of the $IG(a,b)$ is

$$f_X(x)=\sqrt{\frac{b}{2\pi x^3}}\exp\{-\frac{b(x-a)^2}{2a^2x}\},\ x>0,$$

and its cumulative distribution function (cdf) is given by

$$F(x|a,b)=\Phi[\sqrt{\frac{b}{x}}(\frac{x}{a}-1)]+\exp(\frac{2b}{a})\Phi[-\sqrt{\frac{b}{x}}(\frac{x}{a}+1)], \tag{1}$$

where $\Phi(.)$ is the standard normal cdf. Following convention, we let $\Lambda(0)=0$ and $X(0)=0$, and thus $X(t)$ follows $IG(\mu\Lambda(t),\nu(\Lambda(t))^2)$.

In many engineering applications, the failure time T for an item is defined as the time at which the degradation path first reaches a predetermined threshold d. Consider the IG process $\{X(t),\ \ t\geq 0\}$ with $X(t)\sim IG(\mu\Lambda(t),\nu(\Lambda(t))^2)$. Because of the monotonicity property of the IG process, the cdf of T can be readily obtained as following

$$\begin{aligned}P(T\leq t)\ &=P(X(t)\geq d)=1-P(X(t)<d)\\ &=1-\left(\Phi[\sqrt{\frac{\nu}{d}}(\frac{d}{\mu}-\Lambda(t))]+\exp(\frac{2\nu\Lambda(t)}{\mu})\Phi[\sqrt{\frac{\nu}{d}}(\frac{d}{\mu}+\Lambda(t))]\right)\end{aligned} \tag{2}$$

(Ye and Chen [21] and Wang and Xu [18]).

In this paper, we assume that the degradations of the components are dependent and use a copula structure to model the dependency. A copula is the function that connects the joint distribution with individual marginal distribution functions. Let $\mathbf{X}=(X_1,X_2,...,X_p)^T$ be a p-dimensional random vector with marginal cdfs $F_1(x_1)$, $F_2(x_2)$, ..., $F_p(x_p)$ and H be their joint cdf. According to Sklars theorem [11], there exists a unique copula $C(.)$ such that, for all $x_1,x_2,...,x_p$ in R,

$$H(x_1,x_2,...,x_p)\ =\ C(F_1(x_1),F_2(x_2),...,F_p(x_p)) \tag{3}$$

It states any multivariate distribution can be decomposed into a copula and its marginals. Thus, copula functions offers a much more flexible method to study multivariate distributions. Due to different construction routes, there are three commonly used classes of copulas- Elliptical copulas, Archimedean copulas, and extreme-value copulas. Among these copulas, Archimedean copulas have a wide application because they can be constructed easily and can be extended from 2-dimension to p-dimension when some conditions are satisfied [8]. One of the popular Archimedean copulas is the so-called Frank copula. For bivariate case, it is given by

$$C_{\lambda}(u,v) = -\frac{1}{\lambda}\ln\left\{1+\frac{(\exp(-\lambda u)-1)(\exp(-\lambda v)-1)}{\exp(-\lambda)-1}\right\},$$

where $\lambda \in (-\infty,\infty) \setminus \{0\}$ is an association parameter, which is used to measure the dependency between two variables.

Consider a degrading complex k-out-of-n:G system, subject to degradation over time. The system contains n elements such that each element has two components. Suppose that elements work independently, but the degradation of both components of the ith element, $X_i^1(t)$ and $X_i^2(t)$, are dependent stochastic processes for $i = 1,2,...,n$. The following assumptions are considered throughout the paper to study such systems.

1) All components have increasing degradation paths and the hth component fails when $X^h(t)$ reaches or exceeds the given threshold value d_h for $h = 1,2$.

2) Degradations of hth component is distributed with the IG process with the cdf $F_{X^h(t)}(.;\theta_h)$ and the pdf $f_{X^h(t)}(.;\theta_h))$ at time t for $h = 1,2$, where θ_h stands for the parameters of the hth degradation process in each element.

3) The degradation of two components are dependent to each other by copula $C_{\lambda}(.)$ as follows

$$\begin{aligned} &F_{X^1(t),X^2(t)}(d_1,d_2)\\ &= C_{\lambda}(F_{X^1(t)}(d_1;\theta_1),F_{X^2(t)}(d_2;\theta_2)) \end{aligned}$$

$$= -\frac{1}{\lambda} \ln \left\{ 1 + \frac{(\exp(-\lambda F_{X^1(t)}(d_1;\theta_1)) - 1)(\exp(-\lambda F_{X^2(t)}(d_2;\theta_2)) - 1)}{\exp(-\lambda) - 1} \right\} \quad (4)$$

where λ is dependence parameter and $F_{X^h(t)}(.;\theta_h)$ for $h = 1,2$ can be obtained from (2).

3 Degrading complex k-out-of-n:G system

Consider a system consisting of n elements such that each element have two dependent components. In such a system, we assume that the elements are independent and identically distributed. Each element can be represented by a series or parallel structure, whereas the system has a k-out-of-n:G structure, which functions if and only if at least k of its elements function. Suppose all components have increasing degradation paths over time. Typically, we define a vector $\mathbf{X}(t) = (X^1(t), X^2(t))$ to indicate the performance measurement for each element at time t. Further denote a vector $\mathbf{d} = (d_1, d_2)$, representing the failure threshold; i.e. if $X^h(t) \geq d_h$, the hth component is considered to be failed for $h = 1,2$. Denote the failure time of the hth component by T^h, $h = 1,2$. So using (2), the reliability function for the hth component in each element, for $h = 1,2$ is calculated as

$$R^h(t) = P(T^h > t) = P(X^h(t) < d_h).$$

Note that, when the components in each element are series, the element reliability is given by

$$R(t) = \prod_{h=1}^{2} P(X^h(t) < d_h) = R^1(t)\, R^2(t),$$

if all components are independent with each other. Similarly, for parallel components, the element reliability is given by

$$R(t) = 1 - \prod_{h=1}^{2} (1 - P(X^h(t) < d_h)) = R^1(t) + R^2(t) - R^1(t)\, R^2(t).$$

But, when the component are dependent, we use a copula based model to describe the reliability of complex k-out-of-n systems. First of all note that when two components of each element have a series structure, then the reliability of the elements is given by

$$R(t) = P(T > t) = P(T^1 > t, T^2 > t) = C_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2)). \quad (5)$$

Similarly, when the components of an element have a parallel structure, then the reliability of that element is derived as

$$\begin{aligned} R(t) = P(T > t) &= 1 - P(T \leq t) = 1 - P(T^1 \leq t, T^2 \leq t) \\ &= 1 - \bar{C}_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2)). \quad (6) \end{aligned}$$

Here, the marginal reliability in Equation (5) and (6) can be obtained from (2).

4 MRL function of the complex k-out-of-n:G system with intact components at time t

In this section, the MRL function of a complex k-out-of-n:G system is studied in two cases, whether the components in each element are connected in series or parallel. By definition, the life time of such a system is defined as $T_{n-k+1:n}$. The MRL function of the k-out-of-n:G system with intact components at time t is defined as

$$\Phi_{k:n}(t) = E(T_{n-k+1:n} - t | T_{1:n} > t) = E(T^t_{n-k+1:n}),$$

where $T^t_{n-k+1:n}$ is a conditional random variable defined as

$$T^t_{n-k+1:n} = \{T_{n-k+1:n} - t | \text{ non of the components has failed at time } t\}.$$

4.1 MRL function of the complex k-out-of-n:G series system

In a complex k-out-of-n:G series system, the components in each element are connected in series. Thus, the failure time of a element in such a system is minimum failure time of two components. In other words, the first component failure leads to the failure of the element. To calculate $\Phi_{k:n}(t)$, we need the survival function of conditional random variable $T^t_{n-k+1:n}$.

Lemma 1: The survival function of conditional random variable $T^t_{k:n}$ for a k-out-of-n:G series system is given by

$$
\begin{aligned}
&P(T^t_{n-k+1:n} > t') \\
&= [C_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2))]^{-n} \\
&\times \Bigg\{ \sum_{i=n-k+1}^{n} \binom{n}{i} [C_\lambda(F_{X^1(t+t')}(d_1;\theta_1), F_{X^2(t+t')}(d_2;\theta_2))]^i \\
&[C_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2)) - C_\lambda(F_{X^1(t+t')}(d_1;\theta_1), F_{X^2(t+t')}(d_2;\theta_2))]^{n-i} \Bigg\}.
\end{aligned}
\tag{7}
$$

Proof : Let us denote the lifetimes of the elements of a complex system by $T_1, ..., T_n$, which are independent and identically distributed random variables. By definition , we have

$$
\begin{aligned}
&P(T^t_{n-k+1:n} > t') \\
&= P(T_{n-k+1:n} - t > t' | T_{1:n} > t) \\
&= \frac{P(T_{n-k+1:n} - t > t', T_{1:n} > t)}{P(T_{1:n} > t)} \\
&= P(T_1 > t)^{-n} \sum_{i=n-k+1}^{n} \binom{n}{i} P(\text{exactly } i \text{ of } T's \text{ are} > t+t', T_1 > t, ..., T_n > t) \\
&= P(T_1 > t)^{-n} \sum_{i=n-k+1}^{n} \binom{n}{i} (P(T_1 > t+t'))^i \, (P(t < T_1 < t+t'))^{n-i} \\
&= P(T_1 > t)^{-n} \sum_{i=n-k+1}^{n} \binom{n}{i} (P(T_1 > t+t'))^i \, (P(T_1 > t) - P(T_1 > t+t'))^{n-i},
\end{aligned}
$$

where $P(T_1 > t)$ can de obtained from (5). Hence, the proof is complete.

Therefore, the MRL function of the complex k-out-of-n:G series system by using lemma 1, is given as

$$\Phi_{k:n}(t)$$

$$= [C_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2))]^{-n}$$
$$\times \left\{ \sum_{i=n-k+1}^{n} \binom{n}{i} \int_0^\infty [C_\lambda(F_{X^1(t+t')}(d_1;\theta_1), F_{X^2(t+t')}(d_2;\theta_2))]^i \right.$$
$$\left. [C_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2)) - C_\lambda(F_{X^1(t+t')}(d_1;\theta_1), F_{X^2(t+t')}(d_2;\theta_2))]^{n-i} dt' \right\}. \quad (8)$$

4.2 MRL function of the complex k-out-of-n:G parallel system

The complex k-out-of-n:G parallel system is another complex system, where components are connected in parallel. The failure time of a parallel element is the maximum failure time of the components.

Lemma 2: The survival function of conditional random variable $T^t_{n-k+1:n}$ for a k-out-of-n:G parallel system is given by

$$P(T^t_{n-k+1:n} > t')$$
$$= [1 - \bar{C}_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2))]^{-n}$$
$$\times \left\{ \sum_{i=n-k+1}^{n} \binom{n}{i} [1 - \bar{C}_\lambda(F_{X^1(t+t')}(d_1;\theta_1), F_{X^2(t+t')}(d_2;\theta_2))]^i \right.$$
$$\left. [\bar{C}_\lambda(F_{X^1(t+t')}(d_1;\theta_1), F_{X^2(t+t')}(d_2;\theta_2)) - \bar{C}_\lambda(F_{X^1(t)}(d_1;\theta_1), F_{X^2(t)}(d_2;\theta_2))]^{n-i} \right\}. \quad (9)$$

Using Lemma 2, the MRL function of the k-out-of-n:G parallel system with intact components at time t, is given by

$$E(T^t_{n-k+1:n}) = \int_0^\infty P(T^t_{n-k+1:n} > t')dt',$$

where $P(T^t_{n-k+1:n} > t')$ is calculated in (9).

5 Sensitive analysis

In this section, the sensitivity of the MRL function of a complex 1-out-of-6:G series system is provided for different values of t and λ. To do this, let a power transformation on the time scale such that $\Lambda(t;\gamma_h) = t^{\gamma_h}$ for $h = 1,2$. Also, assume the Frank copula with parameter λ as the dependence structure of the two components of each element. Thus, the model parameters are given as $\theta_1 = (\nu_1, \mu_1, \gamma_1)$, $\theta_2 = (\nu_2, \mu_2, \gamma_2)$ and λ. Here, we consider the parameters of marginal degradation processes as $\theta_1 = (1,2,0.5)$, $\theta_2 = (2,2,0.4)$ and $\lambda = 2$. Also, we assume that the degradation thresholds are $d_1 = 3$ and $d_2 = 4$. Figure 1 (the left hand plot) that the MRL function of a complex 1-out-of-6:G series system increases for $0 < t < 7.41$, and decreases for $t > 7.41$. Further, to demonstrate how the dependence parameter effects the MRL function, we provide the plot of the MRL function with respect to λ at a fixed point $t = 12$ in Figure 1 (the right hand plot). It is observed that the MRL function increases with respect to λ.

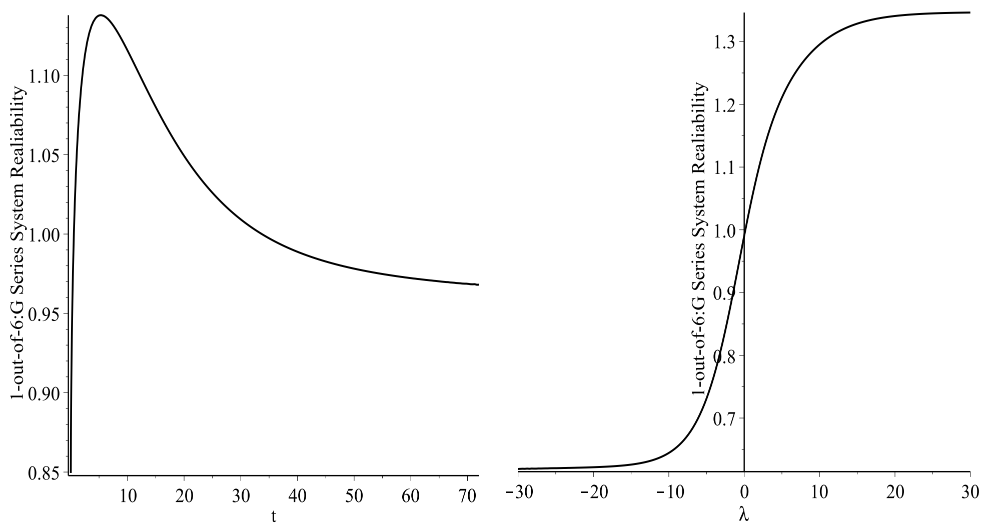

Figure 1: The MRL function of the complex 1-out-of-6:G series system w.r.t t and λ.

6 Conclusion

We have studied the MRL function for a degrading complex k-out-of-n:G system consisting of n elements, each having two dependent components under degradation performance. We assumed that degradation paths come from the IG process and the dependency between components was modeled by Frank copula. A sensitivity analysis was done and the behavior of the MRL function in a special example of complex systems was studied. The results of this research are in progress. Some other copulas may be considered to model the dependency of the components. Also, the MRL function of the complex systems may also be investigated under general path degradation model.

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Quantile Residual Life Estimator in an Ordered Context

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Abstract: In this paper we propose estimator of quantile residual life for two ordered quantile residual life. It has been shown that this estimator is strongly uniformly consistent and asymptotically unbiased. Simulation results indicate that both of the restricted estimators improve on the empirical (unrestricted) estimators in terms of mean squared error, uniformly at all quantiles, and for a variety of distributions.

Keywords: Empirical estimator, Quantile residual life, Stochastic order.

1 Introduction

Let T_1 and T_2 be random variables representing the lifetimes of two populations. These could be patients undergoing two different treatments or the times to recurrence of cancer after the patients have been treated with different kinds of therapies. In the industrial engineering context, T_1 and T_2 could represent the lifetimes of two different brands of an appliance. Suppose that we are confronted with the problem of comparing these two populations to see which one has longer life. A naive approach would be to just compare the means of the two random variables. Rather than basing the decision on two single points, one could compare T_1 and T_2 under a stochastic ordering restriction.

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The mean residual life (MRL) function that is associated with T is defined as

$$m(t) = \begin{cases} E(T-t|T \geq t) & t < u \\ 0 & t \geq u, \end{cases}$$

when the expectation above exists. Comparisons of two random variables in terms of their mean residual life functions leads to the *mean residual life order*. This ordering is weaker than the hazard rate ordering but it is stronger than the variability ordering. [1] is a reference text in this topic.

Suppose that M_1 and M_2 are two MRL functions, e.g., those corresponding to the control and the experimental groups in a clinical trial. It may be reasonable to assume that the remaining life expectancy for the experimental group is higher than that of the control group at all times in the future, i.e., $M_1(t) \leq M_2(t)$ for all t. However, randomness of data will frequently show reversals of this order restriction in the empirical observations. [2] presented two estimators of the MRL subject to such an order restriction.

The main contribution of this paper is to propose two estimators of two (quantile residual life) QRL functions under an order restriction similarly to what [2] did for the MRL functions. The advantage of our estimators is that they are based on the QRL function instead of the MRL: The MRL function sometimes has weaknesses that may prevent its use ([3]). For example, the mean residual life function may not exist. Or, even when it does it may have some practical shortcomings, especially in situations where the data are censored, or when the underlying distribution is skewed or heavy-tailed. In such cases, either the empirical mean residual life function cannot be calculated, or a single long-term survivor can have a marked effect upon it which will tend to be unstable due to its strong dependence on very long durations. Besides, in an experiment it is often impossible or impractical to wait until all items have failed. For those reasons, it is more convenient to consider the median, or any other quantiles, of the residual life instead, since it is less sensitive to outliers or censored data.

The α-QRL function associated to T, is the $alpha$-quantile of $T_t = T|T \geq t$ and can be expressed as

$$q_\alpha(t) = \inf\{x: \ R(t+x) \leq \bar{\alpha}R(t)\}, \tag{1}$$

with $\bar{\alpha} = 1-\alpha$. When R is a continuous function, inequality in (1) reduces to equality and, in a more simple way, the previous expression can be rewritten as

$$q_\alpha(t) = R^{-1}(\bar{\alpha}R(t)) - t,$$

where $R^{-1}(p) = \inf\{x: \ R(t) \leq p\}$.

[3] and [4] explained in detail the potential advantages of the median residual lifetime over the mean residual lifetime and, recently, [5] extended the quantile residual life function to the multivariate context.

The α-quantile residual life function, denoted by q_α, is defined for any $t < u$ by letting $q_\alpha(t)$ be the α-quantile of T_t, $0 < \alpha < 1$. In [6] a family of stochastic orders of random variables defined via the comparison of their quantile residual life functions was introduced. The need for a tool to compare the α-QRLs related to different populations has been widely motivated in [7] and [8].

This paper is organized as follows. The estimators of two ordered quantile residual life functions are proposed in Section 2. These estimators are shown to be strongly uniformly consistent and asymptotically unbiased in Section 3. In order to analyse the bias and the mean squared error of the new estimators, a simulation simulation study has been carried out. The results are presented in Section 4. Finally, in Section 5 we illustrate the behaviour of our estimator with a real data example and in Section 7 the main conclusions are derived.

2 The estimators

Let T_1 and T_2 be random variables representing the lifetimes of two populations with reliability functions R_1 and R_2 and α-QRL functions $q_{\alpha,1}$ and $q_{\alpha,2}$, respectively. Suppose that we are confronted with the problem of comparing these populations to see which one has longer life based on a particular α-QRL function, $\alpha \in (0,1)$. In this section we describe the construction of estimators of two ordered quantile residual life functions based on the empirical estimator whose definition we recall below.

Let $\{T_1, \cdots, T_n\}$ be an independent and identically distributed sample of T. A natural empirical counterpart of q_α is the sample α-quantile residual life function, which is given by

$$\widehat{q}_\alpha(t) = \widehat{F}^{-1}(\alpha + (1-\alpha)\widehat{F}(t)) - t, \quad t < T_n, \tag{2}$$

where $\widehat{F}$ denotes the empirical cumulative distribution function constructed from the sample and T_n is the largest order statistic. For $t \geq T_n$, $\widehat{q}_\alpha(t) = 0$.

Note that $\widehat{q}_\alpha$ is a piecewise linear function with jump discontinuities. It consists of line segments with slope equal to -1 with jump discontinuities. The estimator in (2) was studied by [9]. Further properties were provided by [10], [12], [13], and [11].

In order to explain the construction of the proposed estimators we must differentiate between two scenarios. The first one in which one of the QRL functions is known and then when the two of them are unknown.

2.1 The 1-sample case

Suppose that $q_{\alpha,2}$ is known and $q_{\alpha,1}(t) \leq q_{\alpha,2}(t)$ for all x. Then, the restricted estimator of $q_{\alpha,1}$ is defined as

$$\widehat{q}^*_{\alpha,1}(t) = \widehat{q}_{\alpha,1}(t) \wedge q_{\alpha,2}(t), \tag{3}$$

where $\widehat{q}_{\alpha,1}$ is the estimator given in (2) for T_1 given the ordered sample

$$\{T_{11}, T_{12}, ..., T_{1n_1}\}.$$

Now, suppose that the order restriction is only on an interval, i.e., $q_{\alpha,1}(t) \leq q_{\alpha,2}(t)$ on $[t_1, t_2)$. The quantile residual life function cannot have a jump down and it should satisfy $q'_{\alpha}(t) \geq -1$ where it exists, so we can define the estimator of $q_{\alpha,1}$, as follows. If $\widehat{q}_{\alpha,1}(t_1) \leq q_{\alpha,2}(t_1)$, we define

$$\widehat{q}^{*}_{\alpha,1}(t) = \begin{cases} \widehat{q}_{\alpha,1}(t), & t < t_1 \\ \widehat{q}_{\alpha,1}(t) \wedge q_{\alpha,2}(t), & t_1 \leq t < t_2 \\ \widehat{q}_{\alpha,1}(t), & t \geq t_2, \end{cases}$$

and if $\widehat{q}_{\alpha,1}(t_1) > q_{\alpha,2}(t_1)$, we define

$$\widehat{q}^{*}_{\alpha,1}(t) = \begin{cases} \widehat{q}_{\alpha,1}(t), & t < c \\ q_{\alpha,2}(t_1) + (t_1 - t), & c \leq t \leq t_1 \\ \widehat{q}_{\alpha,1}(t) \wedge q_{\alpha,2}(t), & t_1 \leq t < t_2 \\ \widehat{q}_{\alpha,1}(t), & t \geq t_2, \end{cases}$$

where $c = \max\{T_i : T_i < t_1\}$ and $c = 0$ when $t_1 \leq T_{11}$. See Figure 1.

Similarly, with the restriction $q_{\alpha,1}(t) \geq q_{\alpha,2}(t)$ (reversed direction), when $\widehat{q}_{\alpha,1}(t_2) \geq q_{\alpha,2}(t_2)$, the estimator of $q_{\alpha,1}(t)$ is defined by

$$\widehat{q}^{*}_{\alpha,1}(t) = \begin{cases} \widehat{q}_{\alpha,1}(t), & t < t_1 \\ \widehat{q}_{\alpha,1}(t) \vee q_{\alpha,2}(t), & t_1 \leq t < t_2 \\ \widehat{q}_{\alpha,1}(t), & t \geq t_2, \end{cases}$$

and when $\widehat{q}_{\alpha,1}(t_2) < q_{\alpha,2}(t_2)$, the estimator is defined to be

$$\widehat{q}^{*}_{\alpha,1}(t) = \begin{cases} \widehat{q}_{\alpha,1}(t), & t < t_1 \\ \widehat{q}_{\alpha,1}(t) \vee q_{\alpha,2}(t), & t_1 \leq t < t_2 \\ q_{\alpha,2}(t_2) + (t_2 - t), & t_2 \leq t < c \\ \widehat{q}_{\alpha,1}(t), & t \geq c, \end{cases}$$

where $c = \min\{T_i : T_i > t_2\}$.

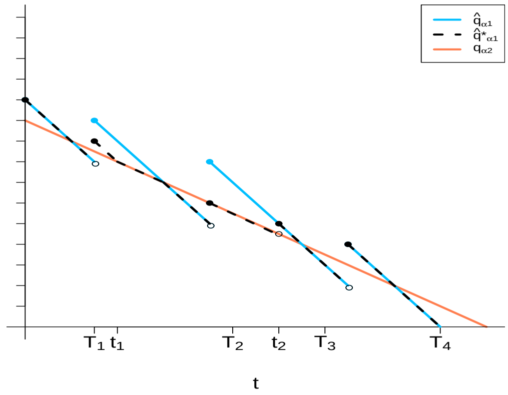

Figure 1: Illustration of $\widehat{q}^*_{\alpha,1}$ in the one sample context

2.2 The 2-sample case

In this context we have two samples $\{T_{11}, T_{12}, ..., T_{1n_1}\}$ and $\{T_{21}, T_{22}, ..., T_{2n_2}\}$. Now, assume that the second sample has derived from a population with greater α-QRL function, i.e., $q_{\alpha,1}(t) \leq q_{\alpha,2}(t)$ for all t. Here, we use the same approach as the non parametric maximum likelihood estimator for two stochastically ordered unknown survival functions as [2]. See references therein. First, we estimate the common α-QRL, $q_\alpha(t)$, by pooling two samples and then estimate each of $q_{\alpha,1}(t)$ and $q_{\alpha,2}(t)$ by proper ordering restrictions. The formulas have been given only for the case where $q_{\alpha,1}(t) \leq q_{\alpha,2}(t)$ everywhere. The case of order restriction on an interval only, can be done exactly as in the 1-sample case since our estimation procedure reduces to two separate 1-sample cases.

The common reliability function can be estimated by

$$\widehat{R}(t) = \frac{n_1\widehat{R}_1(t) + n_2\widehat{R}_2(t)}{n_1 + n_2},$$

where $\widehat{R}_i(t)$ is the empirical reliability (survival) function of the i^{th} sample and $n_i\widehat{R}_i(t) = \sum_{j=1}^{n_i} I(T_{ij} > t)$. By Proposition 1, we know that

$$\widehat{q}_\alpha(t) = \widehat{w}_1(t)\widehat{q}_{\alpha,1}(t) + \widehat{w}_2(t)\widehat{q}_{\alpha,2}(t)$$

where $\widehat{q}_{\alpha,i}(t) = \widehat{R}_i^{-1}(\bar{\alpha}\widehat{R}_i(t))$,

$$\widehat{w}_1(t) = \frac{\widehat{R}_2^{-1}(\bar{\alpha}\widehat{R}_2(t)) - \widehat{R}^{-1}(\bar{\alpha}\widehat{R}(t))}{\widehat{R}_2^{-1}(\bar{\alpha}\widehat{R}_2(t)) - \widehat{R}_1^{-1}(\bar{\alpha}\widehat{R}_1(t))},$$

and $\widehat{w}_2(t) = 1 - \widehat{w}_1(t)$. Clearly, $0 \leq \widehat{w}_1(t), \widehat{w}_2(t) \leq 1$. Then, the restricted estimator of $q_{\alpha,1}(t)$ under restriction $q_{\alpha,1}(t) \leq q_{\alpha,2}(t)$ is defined and simplified below.

$$\begin{aligned}\widehat{q}^*_{\alpha,1}(t) &= \widehat{q}_{\alpha,1}(t) \wedge \widehat{q}_{\alpha}(t) \\ &= \widehat{w}_1(t)\widehat{q}_{\alpha,1}(t) + \widehat{w}_2(t)\Big(\widehat{q}_{\alpha,1}(t) \wedge \widehat{q}_{\alpha,2}(t)\Big) \\ &= \widehat{q}_{\alpha,1}(t) - \widehat{w}_2(t)\Big(\widehat{q}_{\alpha,1}(t) - \widehat{q}_{\alpha,2}(t)\Big)I(\widehat{q}_{\alpha,1}(t) > \widehat{q}_{\alpha,2}(t)).\end{aligned} \tag{4}$$

Similarly, the restricted estimator of $q_{\alpha,2}(t)$ under this restriction is

$$\begin{aligned}\widehat{q}^*_{\alpha,2}(t) &= \widehat{q}_{\alpha,2}(t) \vee \widehat{q}_{\alpha}(t) \\ &= \widehat{w}_2(t)\widehat{q}_{\alpha,2}(t) + \widehat{w}_1(t)\Big(\widehat{q}_{\alpha,1}(t) \vee \widehat{q}_{\alpha,2}(t)\Big) \\ &= \widehat{q}_{\alpha,2}(t) + \widehat{w}_1(t)\Big(\widehat{q}_{\alpha,1}(t) - \widehat{q}_{\alpha,2}(t)\Big)I(\widehat{q}_{\alpha,1}(t) > \widehat{q}_{\alpha,2}(t)).\end{aligned} \tag{5}$$

The 1-sample estimator in (3) can be seen as the limit of the estimator in (5) as $n_2 \to \infty$.

Figure 2 is included for illustration purpose.

3 Properties of the estimators

The following results show the strong uniform consistency and the asymptotic unbiasedness of the estimator.

Proposition 1. The estimators $\widehat{q}^*_{\alpha,1}(t)$ and $\widehat{q}^*_{\alpha,2}(t)$ defined in (4) and (5) are consistent.

Proof. Let $b \leq u$ and $u \leq \infty$ shows the upper bound of the range of T. Then, we have

$$\sup_{0\leq t\leq b} |\widehat{q}^*_{\alpha,1}(t) - q_{\alpha,1}(t)| \leq \sup_{0\leq t\leq b} |\widehat{q}^*_{\alpha,1}(t) - \widehat{q}_{\alpha,1}(t)| + \sup_{0\leq t\leq b} |\widehat{q}_{\alpha,1}(t) - q_{\alpha,1}(t)|.$$

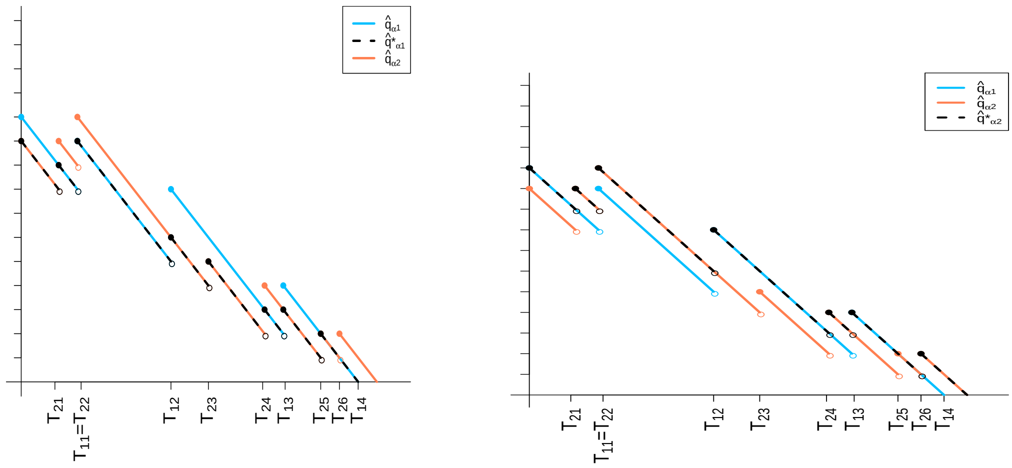

Figure 2: Illustration of $\widehat{q}^*_{\alpha,1}$ ($\widehat{q}^*_{\alpha,2}$) in the two samples context on the left (rigth) panel

The second expression in the right hand side of the inequality, converges to zero, see [örg87]. To show that the first expression also converges to zero, note that

$$\sup_{0\le t\le b} |\widehat{q}^*_{\alpha,1}(t)-\widehat{q}_{\alpha,1}(t)| = \sup_{0\le t\le b} |\widehat{w}_2(t)\Big(\widehat{q}_{\alpha,1}(t)-\widehat{q}_{\alpha,2}(t)\Big)I(\widehat{q}_{\alpha,1}(t)>\widehat{q}_{\alpha,2}(t))|.$$

If $n_1\to\infty$, and n_2 is finite then $\sup_{0\le t\le b}|\widehat{w}_2(t)|\to 0$, so does the whole expression. If $n_1,n_2\to\infty$, then

$$\sup_{0\le t\le b} |\widehat{q}_{\alpha,1}(t)-\widehat{q}_{\alpha,2}(t)|\to 0, \qquad \text{on } \{t: \quad q_{\alpha,1}(t)=q_{\alpha,2}(t)\},$$

and

$$\sup_{0\le t\le b} I(\widehat{q}_{\alpha,1}(t)>\widehat{q}_{\alpha,2}(t))\to 0, \qquad \text{on } \{t: \quad q_{\alpha,1}(t)<q_{\alpha,2}(t)\}.$$

So, $\sup_{0\le t\le b}|\widehat{q}^*_{\alpha,1}(t)-q_{\alpha,1}(t)|\to 0$, i.e., $\widehat{q}^*_{\alpha,1}(t)$ is uniformly consistent. Similarly, we can show that $\widehat{q}^*_{\alpha,2}(t)$ is uniformly consistent. □

Proposition 2. The estimators $\widehat{q}^*_{\alpha,1}(t)$ and $\widehat{q}^*_{\alpha,2}(t)$ defined in (4) and (5) are asymptotically unbiased as $n_1,n_2\to\infty$.

Proof. Let t be fixed. We know that $E(\widehat{q}_{\alpha,1}(t)) \to q_{\alpha,1}(t)$, we refer to [örg87]. Now, let

$$\Delta = \widehat{w}_2(t)\Big(\widehat{q}_{\alpha,1}(t) - \widehat{q}_{\alpha,2}(t)\Big)I(\widehat{q}_{\alpha,1}(t) > \widehat{q}_{\alpha,2}(t)).$$

To show asymptotic unbiasedness, it is sufficient to show that $E(\Delta) \to 0$. By Holder inequality we have

$$E(\Delta) \leq (E(\Delta_1^p))^{\frac{1}{p}}(P(\widehat{q}_{\alpha,1}(t) > \widehat{q}_{\alpha,2}(t)))^{\frac{1}{q}},$$

where $\Delta_1 = \widehat{w}_2(t)(\widehat{q}_{\alpha,1}(t) - \widehat{q}_{\alpha,2}(t))$. But $P(\widehat{q}_{\alpha,1}(t) > \widehat{q}_{\alpha,2}(t))$ converges to zero since convergence almost surely implies convergence in probability. □

4 Simulation

In a simulation study we have analysed the bias and the mean squared error (MSE) of the restricted estimators $\widehat{q}^*_{\alpha,1}(t)$ and $\widehat{q}^*_{\alpha,2}(t)$ we have proposed and compare them to those of the empirical estimators $\widehat{q}_{\alpha,1}(t)$ and $\widehat{q}_{\alpha,2}(t)$, respectively. The quantile α has been set to 0.5, which corresponds to the median residual life function. For the simulations we have considered the Weibull distribution whose reliability function is

$$R(t) = e^{-\theta t^\beta}, \qquad t \geq 0, \theta > 0, \beta > 0,$$

The parameters of the Weibull distributions have been selected such that both increasing and decreasing median residual life functions would be included in the study. The results have been summarized in Tables 1. Every row in each table shows the results for one run in which $r = 5000$ replicates of samples of sizes n_1 and n_2, respectively, were considered. In particular, the results for $n_1 = n_2 = 20$ and 50 are presented. For every two samples $\{T_{11}, T_{12}, ..., T_{1n_1}\}$ and $\{T_{21}, T_{22}, ..., T_{2n_2}\}$ the empirical estimators, $\widehat{q}_{\alpha,1}(t)$ and $\widehat{q}_{\alpha,2}(t)$, and the restricted estimators, $\widehat{q}^*_{\alpha,1}(t)$ and $\widehat{q}^*_{\alpha,2}(t)$, were computed on five deciles (0.1, 0.2, 0.5, 0.8 and 0.9). Then, the bias and the

MSE for these four estimators were calculated. The tables also show the ratio between the MSE for the empirical estimator and the MSE of the restricted one. Although the bias of the unrestricted estimators is lower, the ratio of the MSEs is always larger than 1, which indicates that the restricted estimators outperform the usual ones in terms of the MSE.

Table 1: Simulation results for two Weibull distributions: T_i with parameters (θ_i, β_i) $(i = 1,2)$.

	q	$B(\widehat{q}_{\alpha,1})$	$B(\widehat{q}^*_{\alpha,1})$	$\frac{MSE(\widehat{q}_{\alpha,1})}{MSE(\widehat{q}^*_{\alpha,1})}$	$B(\widehat{q}_{\alpha,2})$	$B(\widehat{q}^*_{\alpha,2})$	$\frac{MSE(\widehat{q}_{\alpha,2})}{MSE(\widehat{q}^*_{\alpha,2})}$
$\theta_1 = 0.015, \beta_1 = 1.2$, $\theta_2 = 0.012, \beta_2 = 1.2$, $n_1 = n_2 = 20$	0.1	-0.21955	-1.18367	1.32724	-0.11778	1.00295	1.19037
	0.2	-0.20173	-1.23694	1.32637	-0.14778	1.04120	1.18599
	0.5	-0.13246	-1.65304	1.48407	-0.08173	1.57764	1.16790
	0.8	-0.33768	-2.84620	1.77574	-0.00086	2.63429	1.06045
	0.9	-0.9894	-4.63811	1.99626	-1.09960	2.25925	1.02442
$\theta_1 = 0.015, \beta_1 = 1.2$, $\theta_2 = 0.012, \beta_2 = 1.2$, $n_1 = n_2 = 50$	0.1	-0.04725	-0.43781	1.20771	-0.22714	0.23333	1.18479
	0.2	-0.04233	-0.48236	1.21101	-0.19048	0.32533	1.18715
	0.5	-0.03008	-0.68932	1.30057	-0.11209	0.69506	1.18851
	0.8	-0.12264	-1.52730	1.54755	0.07010	1.57990	1.12906
	0.9	-0.29556	-2.56233	1.78730	-0.35305	1.80388	1.06816
$\theta_1 = 10, \beta_1 = 0.9$, $\theta_2 = 8, \beta_2 = 0.9$, $n_1 = n_2 = 20$	0.1	0.00042	-0.00274	1.39300	0.00015	0.00345	1.15775
	0.2	0.00071	-0.00298	1.49222	0.00017	0.00391	1.15340
	0.5	0.00026	-0.00517	1.52930	0.00027	0.00607	1.14107
	0.8	0.00015	-0.01066	1.89907	0.00073	0.01171	1.04012
	0.9	-0.00428	-0.01795	1.87934	-0.00369	0.01118	1.01153
$\theta_1 = 10, \beta_1 = 0.9$, $\theta_2 = 8, \beta_2 = 0.9$, $n_1 = n_2 = 50$	0.1	0.00033	-0.00080	1.25667	-0.00038	0.00102	1.15251
	0.2	0.00052	-0.00095	1.26994	-0.00045	0.00130	1.17200
	0.5	0.00030	-0.00211	1.36150	-0.00069	0.00231	1.18272
	0.8	0.00050	-0.00528	1.62210	-0.00039	0.00562	1.11296
	0.9	0.00023	-0.00986	2.07420	0.00055	0.01030	1.06949

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Classical and Bayesian Estimation of Stress-strength Model in the Generalized Linear Failure Rate Distribution when Scale Parameters are Known and Common

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Abstract: In this paper, we study the estimation of $R = P[Y < X]$, also so-called the stress-strength model, when both X and Y are two independent random variables with the generalized linear failure rate distributions when scale parameters are known and common. We address the maximum likelihood estimator (MLE) of R and the associated asymptotic confidence interval. The Bayes estimates of R and the associated credible intervals are also investigated. An extensive computer simulation is implemented to compare the performances of the proposed estimators.

Keywords: Bayes estimator, Generalized linear failure rate distribution, Maximum likelihood estimator.

1 Introduction

The topic of inference on $R = P[Y < X]$ - usually referred to as the stress-strength model has obtained wide attention in the literature, including quality control, engineering statistics, reliability, medicine, psychology, biostatistics, stochastic precedence, and probabilistic mechanical design (see [5], for a comprehensive review). For instance, in a clinical study, Y and X

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can be assumed as the outcomes of a treatment and a control group, respectively, then the following quantity $R = P[Y < X]$ can be considered as the effectiveness of the treatment ([5]). In this case, $(1-R)$ measures the effectiveness of the treatment. Alternatively, for diagnostic tests used to distinguish between diseased and non-diseased patients, the area under the receiver operating characteristics (ROC) curve, based on the sensitivity and the complement to specificity at different cut-off points of the range of possible test values, is equal to R (see [4]).

Another important use of $R = P(Y < X)$ is in reliability contexts, in particular in mechanical reliability of a system, where Y is the strength of a component which is subject to stress X, then R is a measure of system performance, and $(1-R)$ measurs the chance that the system fails. In this situation, the system will fail, if at any time the applied stress is greater than its strength.[5] also present the theoretical and practical results on the theory and applications of the stress-strength relationships in industrial and economic systems.

In reliability context and life science, inferences about R where X and Y are independently distributed are still subject of interest. In this context, the stress-strength model describes the life of a component which has a random strength X and is subjected to random stress Y. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y < X$. Thus $R = P(Y < X)$ is a measure of component reliability.

Estimation of $R = P(Y < X)$, when X and Y are random variables following the specified distributions has been extensively discussed in the literature in both parametric and non-parametric framework. This quantity can be obviously seen as a function of the parameters of the distribution of the random vector (X,Y) and could be calculated in the closed form for a limited number of cases ([5]; [10]; [2]) . For instance, the estimation of R when X and Y are independent and normally distributed has been considered by several authors including [3] , [12] and [4].

[13] reported a list of papers related to the estimation problem of R when X and Y are independent and follow a class of life-time distributions including Exponential, bivariate Exponential, generalized exponential, Gamma distributions, Burr type X model, Weibull distribution, and among others.

The rest of the paper is organized as follows. We briefly introduce the Generalized Linear Failure Rate (GLFR) distribution and study its relevant properties to this study in Section 2. We devote Section 3 to study the estimation of R when the scale parameters of both distributions are common and known. In this section, we derive the ML estimator, Bayes estimators of the stress-strength model, their corresponding confidence or credible intervals and other quantities of interests.

2 Generalized Linear Failure Rate Distribution

It is well known that the exponential, generalized exponential or Rayleigh distribution are among the most commonly used distributions for analyzing lifetime data. These distributions have several desirable properties and nice physical interpretations. They can be used quite effectively in modelling strength and general lifetime data. [6] used different methods to estimate the parameters of the generalized Rayleigh on the observed data. In analyzing lifetime data, the exponential, Rayleigh, linear failure rate or generalized exponential distributions are normally used. It is apparent that the exponential distribution can be only used for the constant hazard function whereas Rayleigh, linear failure rate and generalized exponential distributions can be used for the monotone (increasing in case of Rayleigh or linear failure rate and increasing/ decreasing in case of generalized exponential distribution) hazard functions. In addition, in many practical applications, it is required to apply the non-monotonic function such as bathtub shaped hazard function ([8]). In this paper we use a newly developed distribution by [15] which generalizes the well known exponential distribution, linear failure rate distribution, generalized exponential distribution, and generalized Rayleigh distribution (also known as Burr Type X distribution). They called it *generalized linear failure rate* distribution with three parameters (a,b,α) and denoted by GLFRD(a,b,α). The probability density function (pdf) of $GLFRD(a,b,\beta)$ is given by

$$f_X(a,b,\alpha)(x) = \alpha(a+bx)e^{-(ax+\frac{b}{2}x^2)}(1-e^{-(ax+\frac{b}{2}x^2)})^{\alpha-1} \quad ;\alpha>0 \quad x\geq 0$$

The corresponding cumulative distribution function is as follows

$$F_X(x) = (1-e^{-(ax+\frac{b}{2}x^2)})^{\alpha} \tag{1}$$

where $a,b \geq 0$ are such that $a+b>0$.

This distribution has increasing, decreasing or bathtub shaped hazard rate functions and it also generalizes many well known distributions including the traditional linear failure rate distributions, such as, generalized exponential $(GED(a,\alpha))$ and generalized Rayleigh $(GRD(b,\alpha))$ by putting $b=0$ and $a=0$, respectively.

This distribution is verified to have a decreasing or unimodal pdf. Figure 1 shows some patterns of the pdf of $GLFRD(a,b,\alpha)$, which may have a single mode or no mode at all. In addition, when $\alpha>1$, the hazard rate of

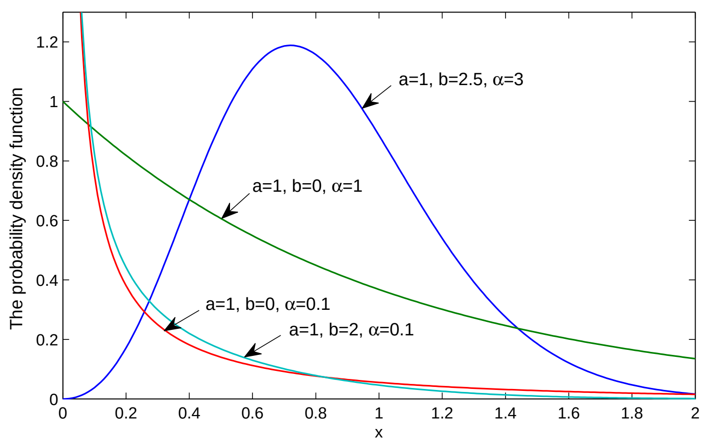

Figure 1: Different shapes of pdf of the GLFR distribution, including unimodal pdf

this distribution is increasing, if $\alpha < 1$, the associated hazard rate is either decreasing if $b = 0$ or inverted bathtub if $b > 0$, and finally when $\alpha = 1$, the hazard rate is either increasing if $b > 0$ or constant if $b = 0$. These patterns are shown in Figure 2 for differen values of the parameters.

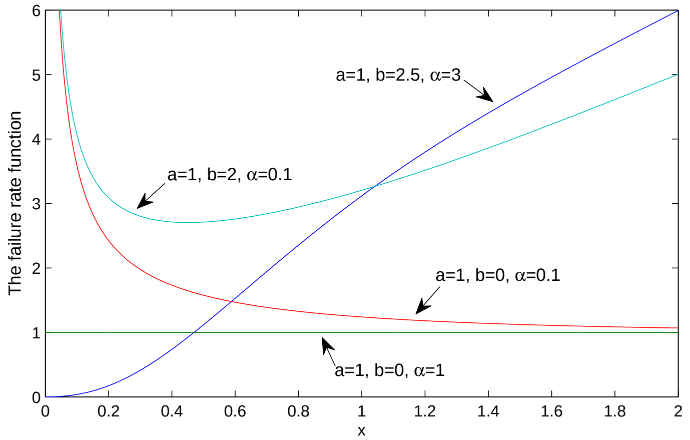

Figure 2: Different shapes of hazard rate function of the GLFR distribution

[14] studied the statistical properties of this distribution and provided some nice physical interpretations. The maximum likelihood estimates (MLEs) of the corresponding parameters appeared to not have the explicit forms, and they can be obtained only by solving two non-linear equations.

3 Estimation of R with known scale parameters

In this section, the main aim is the estimation of $R = P[Y < X]$, where independent random variables X and Y follow the *Generalized Linear Failure Rate* distributions with the known common scale parameters, that is, $X \sim GLFRD(a, b, \alpha)$ and $Y \sim GLFRD(a, b, \beta)$. We wish to derive the MLE of R, its associated confidence intervals, Bayes estimates of R, the corresponding credible interval and study their properties. The stress-strength parameter, R is defined as

$$R = P[Y < X] = \int_0^\infty P(Y < X | X = x) f_X(x) dx$$

$$= \int_0^\infty \alpha(a+bx)e^{-(ax+\frac{b}{2}x^2)}(1-e^{-(ax+\frac{b}{2}x^2)})^{\alpha-1}(1-e^{-(ax+\frac{b}{2}x^2)})^{\beta}dx = \frac{\alpha}{\alpha+\beta} \tag{2}$$

3.1 MLE of R

In this section, we consider the estimation of R when (a,b) are known, and without loss of generality, we assume that $(a,b) = (1,2)$. Let $X_1, X_2, \ldots, X_n$ be a random sample from $GLFR(1,2,\alpha)$ and $Y_1, Y_2, \ldots, Y_m$ be a random sample from $GLFR(1,2,\beta)$. To compute the MLE of R, the corresponding log-likelihood of the observed sample is given by

$$\ell(\alpha,\beta) = n\ln\alpha + \sum_{i=1}^{n}\ln(1+2x_i) + (\alpha-1)\sum_{i=1}^{n}\ln(1-e^{-(x_i+x_i^2)}) - \sum_{i=1}^{n}(x_i+x_i^2)$$

$$+m\ln\beta + \sum_{j=1}^{m}\ln(1+2y_j) + (\beta-1)\sum_{j=1}^{m}\ln(1-e^{-(y_j+y_j^2)}) - \sum_{j=1}^{m}(y_j+y_j^2) \tag{3}$$

The MLEs of (α, β) denoted by ($\hat{\alpha}$, $\hat{\beta}$) can be derived by solving the following equations

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-e^{-(x_i+x_i^2)})$$

$$\frac{\partial \ell}{\partial \beta} = \frac{m}{\beta} + \sum_{j=1}^{m}\ln(1-e^{-(y_j+y_j^2)})$$

Consequently, ($\hat{\alpha}$, $\hat{\beta}$) are given by

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\ln(1-e^{-(x_i+x_i^2)})}$$

$$\hat{\beta} = \frac{-m}{\sum_{j=1}^{m}\ln(1-e^{-(y_j+y_j^2)})}$$

Duo to the invariant property of maximum likelihood estimators, the MLE of R is obtained by replacing α and β by their MLEs in (2) as follows

$$\hat{R} = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$$

Therefore,

$$\hat{R} = \frac{n\sum_{j=1}^{m}\ln(1-e^{-(y_j+y_j^2)})}{n\sum_{j=1}^{m}\ln(1-e^{-(y_j+y_j^2)}) + m\sum_{i=1}^{n}\ln(1-e^{-(x_i+x_i^2)})}$$

It is trivial to show that $-\alpha\ln(1-e^{-(X_i+X_i^2)})$ follows an exponential distribution with mean 1. Therefore, $-2\alpha\sum_{i=1}^{n}\ln(1-e^{-(X_i+X_i^2)}) \sim \chi^2_{(2n)}$ and $-2\beta\sum_{j=1}^{m}\ln(1-e^{-(Y_j+Y_j^2)}) \sim \chi^2_{(2m)}$. So,

$$\hat{R} \sim \frac{1}{1+\frac{\beta}{\alpha}F}$$

or

$$\frac{R}{1-R} \times \frac{1-\hat{R}}{\hat{R}} \sim F,$$

where the random variable F follows a $F_{(2n,2m)}$ distribution with $2n$ and $2m$ degrees of freedom. So, the probability density function (pdf) of $\hat{R}$ is as follows:

$$f_{\hat{R}}(x) = \frac{1}{x^2B(n,m)}(\frac{n\alpha}{m\beta})^n \times \frac{(\frac{1-x}{x})^{n-1}}{(1+\frac{n\alpha}{m\beta}(\frac{1-x}{x}))^{n+m}},$$

where $0 < x < 1$ and $\alpha, \beta > 0$. The $100(1-\gamma)\%$ confidence interval of R can be obtained as

$$[\frac{1}{1+F_{(1-\frac{\gamma}{2};2m,2n)} \times (\frac{1}{\hat{R}}-1)}, \frac{1}{1+F_{(\frac{\gamma}{2};2m,2n)} \times (\frac{1}{\hat{R}}-1)}]$$

where $F_{(\frac{\gamma}{2};2m,2n)}$ and $F_{(1-\frac{\gamma}{2};2m,2n)}$ are the lower and upper $\frac{\gamma}{2}$th percentile points of a F distribution.

3.2 Bayes estimation of R

Let $X \sim GLFR(1,2,\alpha)$ and $Y \sim GLFR(1,2,\beta)$ be independent random variables with cumulative distribution functions $F_X(x \mid \alpha)$ and $F_Y(y \mid \beta)$ given in (1), respectively. By definition, R can be evaluated as a function of the entire parameter $\theta = (\alpha, \beta)$, by the following relation

$$R = R(\theta) = P(X < Y) = \int F_X(t \mid \alpha) f_Y(t \mid \beta) dt$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is a random sample of size n from X and $\mathbf{y} = (y_1, \ldots, y_m)$ is a random sample of size m from Y. Let $\pi(\theta) = \pi(\alpha)\pi(\beta)$ be a prior pdf on (α, β). We consider the Gamma distributions as the prior distributions on α and β, that is, $\alpha \sim Gamma(\gamma_1, \lambda_1)$ and $\beta \sim Gamma(\gamma_2, \lambda_2)$, with the following density function, respectively

$$\pi(\alpha) = \frac{\lambda_1^{\gamma_1}}{\Gamma(\gamma_1)}\alpha^{\gamma_1-1}e^{-\lambda_1\alpha}, \quad \pi(\beta) = \frac{\lambda_2^{\gamma_2}}{\Gamma(\gamma_2)}\beta^{\gamma_2-1}e^{-\lambda_2\beta}, \quad \alpha, \beta > 0 \qquad (4)$$

The posterior distribution of θ via the Bayes rule is given by $\pi(\theta \mid \mathbf{x}, \mathbf{y}) \propto \pi(\theta)L(\theta \mid \mathbf{x}, \mathbf{y})$, where $L(\theta \mid \mathbf{x}, \mathbf{y})$ is the likelihood function for θ based on $(\mathbf{x}, \mathbf{y})$, where its logarithm is given in (3). The posterior distributions of α and β are independent and are given by

$$\alpha|(\mathbf{x}, \mathbf{y}) \sim Gamma(\gamma_1 + n, \lambda_1 - T_1)$$

$$\beta|(\mathbf{x}, \mathbf{y}) \sim Gamma(\gamma_2 + m, \lambda_2 - T_2)$$

where $T_1 = \sum_{i=1}^{n} \log(1 - e^{-(x_i + x_i^2)})$ and $T_2 = \sum_{j=1}^{m} \log(1 - e^{-(y_j + y_j^2)})$.

Bayesian inference on R is based on the derivation of the posterior pdf of R, which can be obtained using a suitable one-to-one transformation of $\theta = (\alpha, \beta)$ of the form $G : \theta \to (R, \eta)$, with inverse $V = G^{-1}$, and $\eta = \alpha + \beta$. Then, the joint posterior pdf of (R, η) is given by $\pi(R, \eta \mid \mathbf{x}, \mathbf{y}) = \pi(V(R, \eta) \mid \mathbf{x}, \mathbf{y})|J_V(R, \eta)|$, where $|J_V(R, \eta)|$ is the Jacobian of the transformation V, so that

$$\pi_R(r \mid \mathbf{x}, \mathbf{y}) = \int \pi(V(r, \eta) \mid \mathbf{x}, \mathbf{y})|J_V(r, \eta)|d\eta = \int \pi(r, \eta \mid \mathbf{x}, \mathbf{y})|d\eta$$

Since a priori α and β are independent, using the prior distributions presented in (4), the joint posterior distribution of (R, η)

$$\pi(r, \eta \mid \mathbf{x}, \mathbf{y}) = C\eta^{\gamma_1+\gamma_2+n+m-1}\exp\{-\eta[r(\lambda_1 - T_1) - (1-r)(\lambda_2 - T_2)]\}r^{\gamma_1+n-1}(1-r)^{\gamma_2+m-1}$$

where

$$C = \frac{(\lambda_1 - T_1)^{\gamma_1+n}(\lambda_2 - T_2)^{\gamma_2+m}}{\Gamma(\gamma_1 + n)\Gamma(\gamma_2 + m)}$$

Then, the marginal posterior density of R is given by

$$f_R(r \mid \mathbf{x},\mathbf{y}) = K\frac{r^{\gamma_1+n-1}(1-r)^{\gamma_2+m-1}}{[(\lambda_1-T_1)r+(\lambda_2-T_2)(1-r)]^{(n+m+\gamma_1+\gamma_2)}} \text{ for } 0<r<1$$

where

$$K = C\times\Gamma(n+m+\gamma_1+\gamma_2)$$

However, there is no close form for the posterior mean or median and the numerical method is required to derive them, but the posterior mode is the root of $\frac{d}{dr}f_R(r \mid \mathbf{x},\mathbf{y}))=0$ and it is unique (see also[13] for the similar reasoning regarding the Generalized Pareto distribution).

The Bayes estimate of R under the squared error loss function, i.e., the posterior mean can be numerically obtained using the numerical method presented in [9] and [1]. This estimate of R denoted by $\hat{R}_B$ is given by

$$\hat{R}_B = \tilde{R}[1+\frac{\tilde{\alpha}\tilde{R}^2(\tilde{\alpha}(n+\gamma_1-1)-\tilde{\beta}(m+\gamma_2-2))}{\tilde{\beta}^2(n+\lambda_1-1)(m+\lambda_2-1)}], \tag{5}$$

where $\tilde{R}=\frac{\tilde{\alpha}}{\tilde{\alpha}+\tilde{\beta}}$, $\tilde{\alpha}=\frac{n+\gamma_1-1}{\lambda_1-T_1}$ and $\tilde{\beta}=\frac{m+\gamma_2-1}{\lambda_2-T_2}$.

4 Simulation Results

In this section, we present some results based on Monte Carlo simulations to compare the performances of the different estimators described in Sections 3. We consider these two cases separately to draw inference about R. We assume that the data are complete and the common scale parameters a,b are also known. In this case, we consider combination of the small sample sizes: m, n = 15, 25 and 50. Without loss of generality, we set $a=1, b=2$. Table 1 illustrates the stress-strength parameter, R, the MLE ($\hat{R}$), the Bayes estimate ($\hat{R}_B$), the confidence interval based on $\hat{R}$ denoted by CI_{MLE}, and its coverage percentage (cp), based on the simulated data from the GLFR distributions with the different values of α and β.

The Bayes estimate of R is computed, using (5), with respect to the given Gamma prior distributions on α and β. It would be quite conventional to

(n,m)	R	$\hat{R}$	$\hat{R}_B$	CI_{MLE}	cp
(15,15)	0.416	0.421	0.412	(0.261,0.581)	0.952
	0.500	0.498	0.5025	(0.282,0.721)	0.956
	0.583	0.583	0.5833	(0.459,0.715)	0.923
	0.666	0.662	0.6664	(0.340,0.976)	0.788
(25,25)	0.416	0.418	0.415	(0.301,0.535)	0.979
	0.500	0.504	0.4976	(0.379,0.629)	0.977
	0.583	0.583	0.5829	(0.443,0.723)	0.972
	0.666	0.662	0.6658	(0.459,0.866)	0.931
(25,25)	0.416	0.419	0.4158	(0.271,0.567)	0.976
	0.500	0.498	0.4992	(0.353,0.642)	0.983
	0.583	0.583	0.583	(0.452,0.713)	0.989
	0.666	0.668	0.6659	(0.526,0.809)	0.984

Table 1: Simulation results and estimation of the parameters when a, b are known from 1000 samples.

use the non-informative prior distributions for α and β. To avoid having the improper posterior distribution, we set the hyper-parameters of the Gamma distributions as $\gamma_1 = \gamma_2 = \lambda_1 = \lambda_2 = 0.0001$ (see [7]). This is trivial to show that the bias and variance of the Bayes estimate would decrease as one could elicit a more informative prior distributions for α and β.

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On Some Closure Properties of α-mixtures

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Abstract: The α-mixture model is a new flexible family of distributions, that includes many existing mixture models as special cases. This paper is an attempt to develop further properties of this family. Some reliability interpretations of the α-mixture of survival functions for different values of α are provided. Results on the closure property of the α-mixture of IFR, IFRA, and NBU distributions are provided. It is shown that a necessary and sufficient condition for an α-mixture to be IFR, IFRA, and NBU is that the mixing distribution is IFR, IFRA, and NBU for $\alpha > 0$. Also, for the 2-component finite α-mixture, using the conditional mixing distribution, it is shown that the α-mixture of DFR (IFR) distributions is DFR (IFR) for $\alpha > 0$ ($\alpha < 0$).

Keywords: Closure property, Failure rate, Mixture models.

1 Introduction

In practice, homogeneous populations can rarely be found. In most areas, including the lifetime, the distribution of the lifetime populations is not homogeneous. This means that all components in the population have not the same distribution. Populations with specific components are usually

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heterogeneous and consist of a different number of sub-populations. For example when components are mixed with two different product lines due to different work shifts, different raw materials, random environment, etc [4]. Obviously, due to the mentioned diversity in the production line, the lifetime distribution of components of one production line is different from another production line and when mixed, they will lead to heterogeneous populations. Ignoring the heterogeneity can lead to fundamental errors in reliability analysis. Mixture models are usually an effective tool for modeling heterogeneity. A new flexible family of distributions, called the α-mixture model, has been recently proposed by [1] that includes many existing mixture models as special cases.

This short communications investigates some further closure properties of α-mixture family based on some important aging concepts reliability engineering. The rest of the paper is organized as follows. In Section 2, the definition of the α-mixture model and some related concepts as well as some new interpretations for the α-mixture will be reviewed. Section 3, is devoted to the study of some closure properties of α-mixtures of IFR, IFRA, and NBU distributions. Before giving the main results of the paper, we need the following definitions:

Definition 1.1.

- A distribution $F(t)$ is said to be increasing failure rate (IFR) if $\log \bar{F}(t)$ is concave.

- A distribution $F(t)$ is said to be increasing failure rate average (IFRA) if $-\log \bar{F}(t)/t$ is non-decreasing.

- A distribution $F(t)$ is said to be new better than used (NBU) if

$$\bar{F}(t_1+t_2) \leq \bar{F}(t_1)\bar{F}(t_2)$$

for all $t_1, t_2 \geq 0$.

2 The α-mixture model

Let T be a lifetime random variable with the α-mixture distribution. We denote survival function (SF), probability distribution function (PDF), and hazard rate of T by $\bar{F}(t,\alpha)$, $f(t,\alpha)$ and $r(t,\alpha)$, respectively. Let Λ be a mixing random variable taking values on $[0,\infty)$ with PDF and CDF, $\pi(\lambda)$ and $\Pi(\lambda)$, respectively. Moreover, assume that $\bar{F}(t|\lambda)$, $f(t|\lambda)$ and $r(t|\lambda)$ refer to the SF, PDF and hazard rate of the random variable $T|\lambda$, respectively. Following [1] consider the α-mixture model as below:

$$\bar{F}(t,\alpha) = \left(\int_0^\infty \bar{F}^\alpha(t|\lambda)\pi(\lambda)d\lambda\right)^{\frac{1}{\alpha}}, \ \alpha \in (-\infty,\infty). \tag{1}$$

Clearly, $\alpha = 1$ yields the ordinary mixture model, and $\alpha = -1$ gives an infinite extension of the harmonic means of SF $\bar{F}(t|\lambda)$.

Properties of α-mixture

The corresponding PDF of (1) is

$$f(t,\alpha) = \left(\int_0^\infty f(t|\lambda)\bar{F}^{\alpha-1}(t|\lambda)\pi(\lambda)d\lambda\right)\left(\int_0^\infty \bar{F}^\alpha(t|\lambda)\pi(\lambda)d\lambda\right)^{\frac{1}{\alpha}-1}. \tag{2}$$

Now, assume that in (1), $\alpha \neq 0$. In this case, the form of the hazard rate of the α-mixture family can be given as follows.

$$\begin{aligned} r(t,\alpha) = \frac{f(t,\alpha)}{\bar{F}(t,\alpha)} &= \frac{\int_0^\infty f(t|\lambda)\bar{F}^{\alpha-1}(t|\lambda)\pi(\lambda)d\lambda}{\int_0^\infty \bar{F}^\alpha(t|\lambda)\pi(\lambda)d\lambda} \\ &= \int_0^\infty r(t|\lambda)\pi_\alpha(\lambda|t)d\lambda, \end{aligned} \tag{3}$$

where

$$\pi_\alpha(\lambda|t) = \frac{\bar{F}^\alpha(t|\lambda)\pi(\lambda)}{\int_0^\infty \bar{F}^\alpha(t|\lambda)\pi(\lambda)d\lambda}. \tag{4}$$

The authors in [6] have stated that $\pi_\alpha(\lambda|t)$ can be considered as the conditional PDF of $\Lambda|T_\alpha \geq t$, where T_α has the SF $\bar{F}^\alpha(t|\lambda)$ for $\alpha > 0$. For $\alpha \leq 0$, for each $t > 0$, the weighted density corresponding to $\pi(\lambda)$ with the weight function $\bar{F}^{\alpha}(t|\lambda)$ is $\pi_{\alpha}(\lambda|t)$.

The case that $\alpha \to 0$ is denoted by $\bar{F}_{gm}(t)$ which is an extension of the geometric mean. In this case, we obtain

$$\bar{F}_{gm}(t) = \lim_{\alpha \to 0} \bar{F}(t,\alpha) = \exp\left(\int_0^{\infty} \log \bar{F}(t|\lambda)\pi(\lambda)d\lambda\right),$$

which implies that the hazard rate of $\bar{F}_{gm}(t)$ is

$$r_{gm}(t) = \int_0^{\infty} r(t|\lambda)\pi(\lambda)d\lambda, \tag{5}$$

where $r(t|\lambda)$ is the hazard rate corresponding to $\bar{F}(t|\lambda)$. This, in turn, implies that the time behavior of the hazard rate of $\bar{F}_{gm}(t)$ depends on the time behavior the hazard rate of $\bar{F}(t|\lambda)$. For example, if $\bar{F}(t|\lambda)$ is IFR (DFR) so is the hazard rate of $\bar{F}_{gm}(t)$.

Finite α-Mixture

The finite α-mixture of SF's $\bar{F}_i$ for $i = 1,2,..,n$, can be expressed as

$$\bar{F}(t,\alpha) = \left[\sum_{i=1}^{n} p_i \bar{F}_i^{\alpha}(t)\right]^{1/\alpha}, \tag{6}$$

having PDF as below:

$$f(t,\alpha) = \left[\sum_{i=1}^{n} p_i f_i(t) \bar{F}_i^{\alpha-1}(t)\right]\left[\sum_{i=1}^{n} p_i \bar{F}_i^{\alpha}(t)\right]^{\frac{1}{\alpha}-1}, \tag{7}$$

where p_i is the mixing proportion such that $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$, for $i \in \{1,2,...,n\}$ (see, [1]).

If $r(t,\alpha)$ and $r_i(t)$ refer to the hazard rate of the finite α-mixture and hazard rate of the i-th component, respectively, then

$$r(t,\alpha) = \frac{f(t,\alpha)}{\bar{F}(t,\alpha)} = \sum_{i=1}^{n} r_i(t) p_i(t), \tag{8}$$

where $p_i(t) = \frac{p_i \bar{F}_i^{\alpha}(t)}{\sum_{i=1}^{n} p_i \bar{F}_i^{\alpha}(t)}$. In particular, for the finite α-mixture (6), the SF and the hazard rate of the geometric mixture, denoted by $\bar{F}_{gm}(t)$ and $r_{gm}(t)$, respectively, can be written as:

$$\bar{F}_{gm}(t) = \prod_{i=1}^{n} \bar{F}_i^{p_i}(t),$$

and

$$r_{gm}(t) = \sum_{i=1}^{n} p_i r_i(t).$$

In the following, we provide some reliability interpretations of the α-mixture family for different values of α.

(i) The case $\alpha \to 0$, with SF $\bar{F}_{gm}(t) = \prod_{i=1}^{n} \bar{F}_i^{p_i}(t)$, can be considered as a generalized proportional hazards (GPH) model. Also, one can see that $\bar{F}_{gm}(t)$ is the reliability function of a series system that consists of n independent components, where the reliability function of the i-th component follows from the PH model with the PH parameter p_i and the baseline SF $\bar{F}_i(t)$, $i = 1, \ldots, n$. Note that, when $p_i = \frac{1}{n}$, we get $\bar{F}_{gm}^n(t) = \prod_{i=1}^{n} \bar{F}_i(t)$, which is the reliability function of an n-components series system, where the i-th component has reliability $\bar{F}_i(t)$.

(ii) The case $\alpha = -1$. Assume that we have a mixed population of two components with SF's $\bar{F}_1(t)$ and $\bar{F}_2(t)$ and proportions p and $1-p$, respectively. Then the SF of a randomly selected from the population component is

$$\bar{F}_m(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t).$$

The randomly selected component is repeatedly tested to its lifetime exceed the specified time t for the first time. In this case, the average number of the components required to achieve the first success is

$$N_m = \frac{1}{\bar{F}_m(t)} = \frac{1}{p\bar{F}_1(t) + (1-p)\bar{F}_2(t)}. \tag{9}$$

The average number of the components with SF $\bar{F}_1(t)$ required to achieve the first success is $N_1 = \frac{1}{\bar{F}_1(t)}$. Similarly, the average number of the components with SF $\bar{F}_2(t)$ required to achieve the first success is $N_2 = \frac{1}{\bar{F}_2(t)}$. Now, if we calculate the arithmetic means of N_1 and N_2 with weights p and $1-p$, respectively, we have

$$N_h = pN_1 + (1-p)N_2 = \frac{p}{\bar{F}_1(t)} + \frac{1-p}{\bar{F}_2(t)}. \tag{10}$$

Consider the α-mixture model (6) with SF's $\bar{F}_i$, $i = 1, 2$.

$$\bar{F}(t,\alpha) = \left[p\bar{F}_1^{\alpha}(t) + (1-p)\bar{F}_2^{\alpha}(t)\right]^{\frac{1}{\alpha}} \tag{11}$$

The right-hand side of (10) is $\frac{1}{\bar{F}(t,-1)}$, which means $N_h = \frac{1}{\bar{F}(t,-1)}$. On other hand, the right-hand side of (9) is $\frac{1}{\bar{F}(t,1)}$, which means $N_m = \frac{1}{\bar{F}(t,1)}$. As we see, the α-mixture model covers these two averages, and by the monotony property of α-mixtures, we have $N_m \leq N_h$.

(iii) The case $\alpha > 0$. Consider the following two different policies for construct an m-component series system [1].

A. In the first policy, to construct an m-component series system of the same type, a component is randomly selected from a set of n components and then drawn m item from it. If the probability of selecting the i-th component be p_i, $i = 1, 2, \ldots, n$, then the reliability function of constructed m-component series system will be equal to

$$\bar{\mathscr{F}}_1(t) = \sum_{i=1}^{n} p_i \bar{F}_i^m(t) = \bar{F}^m(t,m),$$

where $\bar{F}_i(t)$, $i = 1, \ldots, n$, is the SF of the i-th component and $\bar{F}(t,m)$, is the SF of the finite α-mixture with $\alpha = m$.

B. In the second policy, first, we mixed n components and then drawn all of m components randomly from the mixed population. If the proportion and the SF of the i-component in the mixed population be p_i and $\bar{F}_i(t)$, $i = 1, \ldots, n$, respectively, then the reliability function of constructed m-component series system will be equal to

$$\bar{\mathscr{F}}_2(t) = \left(\sum_{i=1}^{n} p_i \bar{F}_i(t)\right)^m = \bar{F}^m(t,1),$$

where $\bar{F}(t,1)$, is the SF of the finite α-mixture with $\alpha = 1$.

The author in [1], by monotone decreasing property of α-mixtures, have shown that $\bar{\mathscr{F}}_2 \leq_{st} \bar{\mathscr{F}}_1$, that means mixtures of a series system with heterogeneous components are less reliable than a series system with homogeneous components. Thus, for building a series system from different types of components, the 'mixing at the system level' is better than the 'mixing at the component level'.

Another interpretation of the α-mixture, which can be found in [6], is as follows.

Assume that we have a mixed population with n sub-populations, where the SF's of the sub-populations in laboratory condition is $\bar{F}_i(t)$, $i = 1,\dots,n$. Suppose that the proportions of the n sub-populations are p_i, $i = 1,\dots,n$. Assume that the severe condition acts on each sub-population uniformly so that the SF of i-th sub-population will change to $\bar{F}_i^{\alpha}(t)$, where $\alpha > 0$. Then, the SF of a randomly selected individual is

$$\bar{F}_s(t,\alpha) = \sum_{i=1}^{n} p_i \bar{F}_i^{\alpha}(t).$$

Assume that the component is shielded from the severe condition to arrive at the laboratory condition. Hence, the reliability of the selected component in the laboratory condition is

$$\bar{F}(t,\alpha) = \left(\sum_{i=1}^{n} p_i \bar{F}_i^{\alpha}(t)\right)^{\frac{1}{\alpha}},$$

which is the SF of the α-mixture model.

We give the following results for the finite α-mixture using the conditional PDF that coincides with the results given in [1]. Let us consider an α-mixture of two SF's $\bar{F}_1(t)$ and $\bar{F}_2(t)$ with hazard rates $r_1(t)$ and $r_2(t)$, respectively. The α-mixture hazard rate in this case is

$$r(t,\alpha) = r_1(t)\frac{p\bar{F}_1^{\alpha}(t)}{p\bar{F}_1^{\alpha}(t) + (1-p)\bar{F}_2^{\alpha}(t)} + r_2(t)\frac{(1-p)\bar{F}_2^{\alpha}(t)}{p\bar{F}_1^{\alpha}(t) + (1-p)\bar{F}_2^{\alpha}(t)}$$

$$= r_1(t)p_\alpha(t) + r_2(t)(1 - p_\alpha(t)), \tag{12}$$

where the time-dependent probabilities are

$$p_\alpha(t) = \frac{p\bar{F}_1^\alpha(t)}{p\bar{F}_1^\alpha(t) + (1-p)\bar{F}_2^\alpha(t)}, \quad (1 - p_\alpha(t)) = \frac{(1-p)\bar{F}_2^\alpha(t)}{p\bar{F}_1^\alpha(t) + (1-p)\bar{F}_2^\alpha(t)}.$$

From this representation we have:

$$\min\{r_1(t), r_2(t)\} \leq r(t,\alpha) \leq \max\{r_1(t), r_2(t)\}.$$

In particular, if $\bar{F}_1 \geq_{hr} \bar{F}_2$, then

$$r_1(t) \leq r(t,\alpha) \leq r_2(t).$$

Also, we can study directly the closure property of the finite α-mixture of two components by differentiating (12) with respect to t as follows:

$$\begin{aligned} r'(t,\alpha) &= r_1'(t)p_\alpha(t) + p_\alpha'(t)r_1(t) + r_2'(t)(1 - p_\alpha(t)) - p_\alpha'(t)r_2(t) \\ &= r_1'(t)p_\alpha(t) + r_2'(t)(1 - p_\alpha(t)) + p_\alpha'(t)(r_1(t) - r_2(t)) \\ &= r_1'(t)p_\alpha(t) + r_2'(t)(1 - p_\alpha(t)) - \alpha p_\alpha(t)(1 - p_\alpha(t))(r_1(t) - r_2(t))^2. \end{aligned}$$

Therefore, as $r_i'(t) \leq 0$, $i = 1,2$, $r'(t,\alpha) \leq 0$ for $\alpha \geq 0$. This means if both distributions are DFR, then the finite α-mixture is also DFR for $\alpha \geq 0$. Similarly, as $r_i'(t) \geq 0$, $i = 1,2$, $r'(t,\alpha) \geq 0$ for $\alpha \leq 0$. This means if both distributions are IFR, then the finite α-mixture is also IFR for $\alpha \leq 0$.

3 Closure properties of α-mixtures

Recently, [1] have extended the famous result on the closure property of the mixture of DFR distributions. It is well-known that the α-mixtures of IFR (IFRA) distributions are IFR (IFRA) for $\alpha < 0$. But what happens when $\alpha > 0$? To answer this question, first, we extend the result in [5] for α-mixture of IFR distributions. Next, we obtain similar closure theorems for IFRA distributions. We extend the result which is presented in [2]. The following lemma and theorems, given in [5], are helpful tools to study the closure properties in the subsequent section.

Theorem 3.1. *(i) If f and g are log concave, so are $f.g, cf$ and f^a for all $a > 0, c \geq 0$.*

(ii) If f is log concave and non-decreasing and ψ is concave (and nonnegative), then the composition $fo\psi$ is log concave.

Theorem 3.2. *Let F be log concave on $R^m \times R^n$. Then $G(x) = \int F(x,y)dy$ is log concave on R^m. (Here dy is Lebesgue measure on R^n)*

Theorem 3.3. *The following conditions are equivalent:*

(i) Λ is IFR(DFR)

(ii) $\bar{\Pi}$ is a Po1ya frequency function of order 2 $(PF2)(RR2)$

(iii) $\bar{\Pi}$ is log concave(log convex).

Lemma 3.4. *If Λ is IFR, then there exists a continuous non-negative non-decreasing concave function ψ on $[0,\infty)$ such that $\psi(S)$ has the same distribution as Λ, where S is distributed as a standard exponential.*

3.1 Closure property under the notion of IFR

The next theorem explores the IFR closure property of α-mixture.

Theorem 3.5. *If Λ be IFR and $\bar{F}(t|\lambda)$ be log concave in (t,λ) and increasing in λ for each fixed $t \geq 0$, then, $\bar{F}(t,\alpha)$ is log concave in t (IFR) for $\alpha > 0$. Conversely, if $\bar{F}(t,\alpha)$ be log concave in t for $\alpha > 0$ whenever $\bar{F}(t|\lambda)$ satisfies the above condition, then Λ is IFR.*

Proof. Let Λ be IFR and $\bar{F}(t|\lambda)$ be log concave in (t,λ) and increasing in λ for each fixed $t \geq 0$. Then, from Lemma 3.4, we have

$$\int_0^\infty \bar{F}^\alpha(t|\lambda)\pi(\lambda)d\lambda = \int_0^\infty \bar{F}^\alpha(t|\psi(s))e^{-s}ds. \tag{13}$$

Since $\bar{F}(t|\lambda)$ is log concave in (t,λ), from Theorem 3.1 (i), $\bar{F}^\alpha(t|\lambda)$ is log concave for $\alpha > 0$. On the other hand, $\bar{F}(t|\lambda)$ is increasing in λ, from Theorem 3.1 (ii), $\bar{F}^\alpha(t|\psi(s))$ is log concave. Hence from Theorem 3.2, relation (13) is log concave in t. Again from Theorem 3.1 (i), since $\alpha > 0$,

$\bar{F}(t,\alpha)$ is log concave in t.
The converse part follows by taking $\bar{F}(t|\lambda) = I(t < \lambda)$. Thus, $I(t < \lambda)$ is log concave in (t,λ) and non-decreasing in λ for each fixed t. From Theorem 3.1 (i) $I^{\alpha}(t < \lambda)$ is log concave for $\alpha > 0$. Hence, by assumption, $\bar{F}(t,\alpha)$ is log concave in t, again from Theorem 3.1 (i), $\bar{F}^{\alpha}(t,\alpha) = \int \pi(\lambda)d\lambda$ is log concave. Thus, according to Theorem 3.3, Λ is IFR. This completes the proof. □

3.2 Closure property under the notion IFRA

The following lemma, given in [3], is a useful tool for the next theorem.

Lemma 3.6. Λ *is IFRA iff*

$$\int h(\theta)d\Pi(\lambda) \le \left\{\int h^{\beta}(\lambda/\beta)d\Pi(\lambda)\right\}^{1/\beta}$$

for all $0 \le \beta \le 1$ *and all nonnegative increasing functions* h.

Now, we can give the following theorem on the IFRA closure property for α-mixture.

Theorem 3.7. *Suppose that the mixing distribution* $\Pi(\lambda)$ *be an IFRA distribution. Also, let* $\bar{F}(t|\lambda)$ *is increasing in* λ *for each* $t \ge 0$ *and satisfies*

$$\bar{F}(\beta t|\beta\lambda) \ge \bar{F}^{\beta}(t|\lambda) \tag{14}$$

for all $0 < \beta < 1$ *and for all* $t \ge 0$ *and* $\lambda \ge 0$. *Then,* $\bar{F}(t,\alpha)$ *is IFRA for* $\alpha > 0$. *Conversely, if* $\bar{F}(t,\alpha)$ *is IFRA whenever* $\bar{F}(t|\lambda)$ *satisfies the above two conditions, then* Λ *is IFRA.*

Proof. The "if" part of the proof follows because from (1) we have

$$\bar{F}(\beta t,\alpha) = \left(\int_0^{\infty} \bar{F}^{\alpha}(\beta t|\lambda)\pi(\lambda)d\lambda\right)^{\frac{1}{\alpha}}.$$

Then, by assumption

$$\bar{F}(\beta t,\alpha) \geq \left(\int_0^\infty \bar{F}^{\alpha\beta}(t|\lambda/\beta)\pi(\lambda)d\lambda\right)^{\frac{1}{\alpha}}$$
$$\geq \left(\left[\int_0^\infty \bar{F}^{\alpha}(t|\lambda)\pi(\lambda)d\lambda\right]^{\beta}\right)^{\frac{1}{\alpha}}$$
$$= \left(\left[\int_0^\infty \bar{F}^{\alpha}(t|\lambda)\pi(\lambda)d\lambda\right]^{\frac{1}{\alpha}}\right)^{\beta} = \bar{F}^{\beta}(t,\alpha),$$

where the last inequality follows from Lemma 3.6. The "only if" part follows by taking $\bar{F}(t|\theta) = I(t < \theta)$. This completes the proof. □

3.3 Closure property under the notion of NBU

To proof an NBU closure result, we need the following lemma from [2].

Lemma 3.8. *Λ is NBU iff*

$$\int g(\beta\lambda)\, h[(1-\beta)\lambda]\ d\Pi(\lambda) \leq \int g(\lambda)\, d\Pi(\lambda)\ \int h(\lambda)\, d\Pi(\lambda)$$

for all nonnegative, increasing functions g and h and all $0 < \beta < 1$.

Theorem 3.9. *Suppose that the mixing distribution $\Pi(\lambda)$ is an NBU distribution. Also, let $\bar{F}(t|\lambda)$ be increasing in λ for each $t \geq 0$ and satisfies*

$$\bar{F}(t|\lambda) \leq \bar{F}(\beta t|\beta\lambda)\bar{F}((1-\beta)t|(1-\beta)\lambda)$$

for all $0 < \beta < 1$ and for all $\lambda \geq 0$ and $t \geq 0$. Then, $\bar{F}(t,\alpha)$ is NBU for $\alpha > 0$. Conversely, if $\bar{F}(t,\alpha)$ is NBU whenever $\bar{F}(t|\lambda)$ satisfies the above two conditions, then Λ is NBU.

Proof. The direct part of the proof follows because from (1) for $0 < \beta < 1$, we have

$$\bar{F}(t,\alpha) = \left(\int_0^\infty \bar{F}^{\alpha}(t|\lambda)\pi(\lambda)d\lambda\right)^{\frac{1}{\alpha}}$$

$$\leq \left(\int_0^\infty \bar{F}^\alpha(\beta t|\beta\lambda)\bar{F}^\alpha((1-\beta)t|(1-\beta)\lambda)\pi(\lambda)d\lambda \right)^{\frac{1}{\alpha}}$$
$$\leq \left(\left(\int_0^\infty \bar{F}^\alpha(\beta t|\lambda)\pi(\lambda)d\lambda \right) \left(\int_0^\infty \bar{F}^\alpha((1-\beta)t|\lambda)\pi(\lambda)d\lambda \right) \right)^{\frac{1}{\alpha}}$$
$$= \bar{F}(\beta t, \alpha)\, \bar{F}((1-\beta)t, \alpha),$$

where the first and second inequality follows from assumption and Lemma 3.8, respectively. The converse part follows by taking $\bar{F}(t|\theta) = I(t < \theta)$. This completes the proof. □

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Effect of Aging and Environmental Shocks on Reliability of Coherent Systems Consisting of Heterogeneous Components

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Abstract: In reliability engineering, a system and its components, in addition to internal failures, are often subject to external lethal shocks. Usually, the random shocks are produced by a random environment and modeled by a stochastic process. In this talk, using the theory of survival signature, a general approach in assessing the reliability of systems with heterogeneous components is presented. We also propose a preventive maintenance model for a multicomponent system whose components are subject to both internal failures and fatal shocks. The criteria that will be optimized are the cost functions formulated based on the repair costs of the components and the whole system.

Keywords: Aging, Lethal shock, Preventive maintenance, Survival signature.

1 Introduction

In real life situation, a system and its components may fail due to various factors. The failure may occur according to aging over time and/or external shock processes. For example, civil infrastructures, such as highway bridges, may fail because of material fatigue or wear, or may be subjected

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to earthquakes or hurricanes. When assessing the reliability of such systems, neglecting the effects of external shocks tends to produce large prediction errors and even false reasoning. As a result, it is of great importance to establish a model for system reliability analysis, which considers both internal failures and external shocks.

In assessing the reliability of systems, a problem of interest, for academic researchers and system designers, is to maintain the system in optimum working conditions. Since almost all types of systems are exposed to failure, the appropriate maintenance of such systems or their components is vital to keep the system in proper working conditions and also to avoid high costs of sudden failure of the system. Generally, two types of maintenance operations may be received by a system that deteriorates with age: corrective maintenance (CM) and preventive maintenance (PM). The CM is performed on a failed system to restore it to operating condition, while the PM is a planned maintenance that is applied on the system before its failure to bring it back to a better working condition.

In recent years, the problem of determining optimal maintenance policies has been extensively studied in the literature. Age-based PM policy is among the first maintenance models in which the system is replaced at age T_{PM} or at system failure, whichever occurs first (see, [2]). [19] presented a review of age replacement models with new perspectives. An age-based PM policy for a binary coherent system consisting of independent exponential components are presented in [8]. There is also an increasing interest in maintaining a system whose components are completely reliable, but are exposed to some shocks from various sources. Based on the notion of signature, ([9],[10]) studied some PM models for a systems whose components are subject to shocks. [4] considered preventive maintenance of systems operating in a random environment modeled by a Poisson process of shocks. Recent developments on maintenance strategies can be found, for example, in [20] and [11] and [13].

The concepts of signature and survival signature of the system are two beneficial tools for assessing the reliability of a coherent system. The notion of signature was originally defined by [15]. The signature can be effectively utilized to compute the system reliability and to compare coherent systems with different structures. Recent development of the signature-based reliability analysis can be found, for example, in [16], [14] and [17]. [5] later introduced the concept of survival signature for systems with components of multiple types. For references on the reliability properties of systems with multiple types of components based on the concept of survival signature, we refer the reader to [6], [1] and [7].

In the present paper, we consider a coherent system with independent but possibly non-identical components. Along with internal failures, the components are assumed to be subject to external shocks as well. This setting provides a more realistic approach to system reliability analysis. Under the given failure scenario, we study the survival signature-based reliability representation of the system lifetime. This representation enables us to propose some cost-based optimal maintenance models such that at the time of system failure or at the PM time, the maintenance actions are performed not only on the entire system but also on each component depending on its status.

The reminder of the paper is arranged as follows. In Section 2, we provide some formulas for evaluating the reliability of a system whose components are exposed to both aging and shocks. Section 3 presents a maintenance policy based on the reliability models developed in Section 2. Extensive graphical and numerical examples to theoretical results appeared in the paper are also presented.

2 The system reliability under shocks and aging

In this section, the reliability of a coherent system (c.f. [3]) consisting of n heterogeneous components is evaluated. Consider an n-component coherent system with $L \geq 2$ types of components consisting n_j components of type $j,\ j = 1,2,...,L$, such that $\sum_{j=1}^{L} n_j = n$. Assume that the lifetimes of all components are independent random variables and the components of type j are identical with a common cumulative distribution function $F_j,\ j = 1,2,...,L$. The survival signature for such a system, denoted by $\Phi(i_1,i_2,...,i_L)$, is defined as the probability that the system functions given that precisely i_j components of type j function, $j = 1,2,...,L$. The survival signature can then be written as (see [5])

$$\Phi(i_1,i_2,...,i_L) = \left(\prod_{j=1}^{L} \binom{n_j}{i_j}^{-1} \right) \sum_{x \in S} \phi(x),$$

where $\phi(x)$ is the structure function of the system.

Now, we present a more general setting in which internal failures and external shocks are two causes of components failures. It is assumed that the components of type $j,\ j = 1,2,...,L$, are exposed to a specific sequence of lethal shocks. Further, we suppose that the shock processes are independent of the internal failures. Let $P_{i_j}^{(j)}(t)$ denotes the probability of arriving i_j shocks to components of type j in $[0,t)$ and $\bar{P}_{n_j}^{(j)}(t) = \sum_{\nu_j=n_j}^{\infty} P_{\nu_j}^{(j)}(t)$. Before proceeding to find the system reliability under the described setting, we present a result which is useful for our derivations. The proof may be found in Tavangar and Hashemi (2021).

Proposition 2.1. *For a coherent system consisting of L types of components (described at the beginning of this section) that are subject to both shocks and aging, the probability that i_j, $i_j \in \{0,1,...,n_j\}$, components of type j, $j = 1,2,...,L$, has been failed up to time $t > 0$ is given as*

$$P\left(N^{(1)}(t) = i_1,...,N^{(L)}(t) = i_L\right) = \prod_{j \in E} \left\{ \sum_{k_j=0}^{n_j} \bar{P}_{n_j-k_j}^{(j)}(t) \binom{n_j}{k_j} F_j^{k_j}(t) \bar{F}_j^{n_j-k_j}(t) \right\}$$

$$\times \prod_{j\in E^c} \left\{ \sum_{k_j=0}^{i_j} P^{(j)}_{i_j-k_j}(t) \binom{n_j}{k_j} F_j^{k_j}(t) \bar{F}_j^{n_j-k_j}(t) \right\}, \tag{1}$$

where $N^{(j)}(t)$ denotes the total number of failed components of type j in $[0,t)$ and $E = \{j : i_j = n_j\} \subseteq \{1,2,...,L\}$.

Using Proposition 2.1, the reliability function of the system subject to both internal failures and external shocks can be expressed as

$$\bar{H}(t) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_L=0}^{n_L} \Psi(i_1,...,i_L) P\left(N^{(1)}(t) = i_1,...,N^{(L)}(t) = i_L\right), \quad t \geq 0, \tag{2}$$

where $\Psi(i_1,i_2,...,i_L) = \Phi(n_1-i_1, n_2-i_2,...,n_L-i_L)$ and $P(N^{(1)}(t) = i_1,..., N^{(L)}(t) = i_L)$ is given in (1).

The application of the above proposed method is presented in the following example.

Example 2.2. Consider the automotive braking system (ABS) consisting of four component types M, H, C and P which is depicted in Figure 1. The system has a master brake cylinder (M), four wheel brake cylinders (C_1–C_4), four braking pads (P_1–P_4) and a hand brake mechanism (H). Such a system is analyzed in [18] who proposed a condition-based maintenance strategy by means of the survival signature. The values of $\Psi(i_1,i_2,i_3,i_4)$ for the ABS is given in Tavangar and Hashemi (2021).

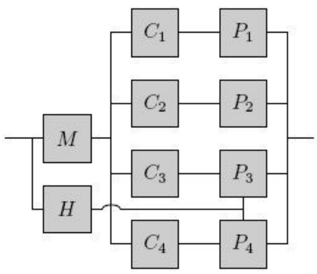

Figure 1: Automotive braking system.

We consider different lifetime distributions and shock processes for each particular type of components. Table 1 summarizes these information for each type. We assume that the components of types M, H, C and P are subject to non-homogeneous Poisson shock processes with different intensity $\lambda(t)$. Also, the lifetimes of components are assumed to have Weibull distributions $W(\alpha,\beta)$ with the reliability function

$$\bar{F}(t) = e^{-(\frac{t}{\beta})^{\alpha}}, \quad t \geq 0 \quad \alpha > 1, \quad \beta > 0.$$

Table 1: Information on failure mechanism of components.

No.	Components	Distribution	$\lambda(t)$
1	M	$W(2,1)$	$0.2\,t^a$
2	H	$W(2,1.5)$	$0.6\,t^a$
3	C_1,C_2,C_3,C_4	$W(2,1.2)$	t^a
4	P_1,P_2,P_3,P_4	$W(2,1.8)$	$1.2\,t^a$

Now with having $\Psi(i_1,i_2,...,i_L)$ and Equation (2), the reliability of the system subject to internal deterioration, lethal shocks, or both internal failures and external shocks can be calculated. The results are shown in Figure 2(a) as the doted line, dashed line and solid line, respectively. It can be observed that the reliability of the system under both aging and shocks is always lesser than that with only one cause of failure. Also, it is seen that the reliability of the system impinged only by shocks is greater than that solely subject to aging before time $t = 2.1$, but later, the situation is reversed.

Figure 2(b) shows the graphs of the hazard rates of the system subject to aging $r_a(t)$, shocks $r_s(t)$ and both aging and shocks $r(t)$. Again, it is observed that the hazard rate $r(t)$ is always higher than $r_a(t)$ and $r_s(t)$, and that $r_a(t)$ is greeter than $r_s(t)$ up to time $t = 2.1$, but afterwards, the situation is reversed.

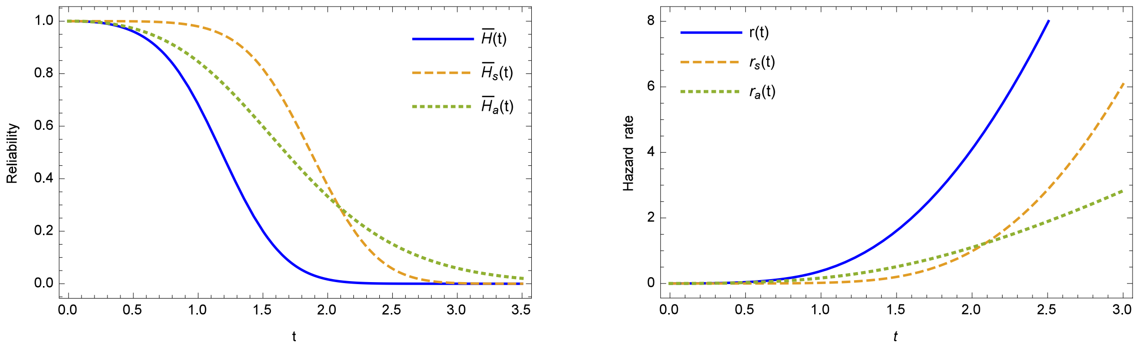

Figure 2: (a) The reliability function of the system. (b) The hazard rate of the system.

3 Preventive maintenance model based on cost criterion

In this section, a PM model is proposed for multi-component coherent systems whose components are exposed to environmental shocks and aging. It is worth noting that the cost function in the proposed model are presented using the notion of survival signature. In fact this concept enables us to introduce some maintenance models in which repair/replacement actions are performed not only on the entire system but also on each component depending on its status.

Consider an original n-component coherent system consisting of L, $2 \leq L \leq n$, types of components where there exists n_ℓ components of type ℓ, $\ell = 1,2,\ldots,L$, and $\sum_{\ell=1}^{L} n_\ell = n$. Assume that this system whose components are subject to external lethal shocks in addition to internal deterioration, starts to function at time $t = 0$. Also, we assume that the components of type ℓ, $\ell = 1,2,...,L$, are exposed to a specific sequence $\{N_s^{(\ell)}(t),\ t \geq 0\}$ of lethal shocks. In this section, we propose a maintenance model for such a system with a decision variable T_{PM} on the time interval $(0,\infty)$. There are two possibilities: The system has failed before time T_{PM} or the system is still operating at that time. If the system fails in the interval $(0,T_{PM})$, the operator decides to perform a CM on the whole system with cost c_r. However, if the system is working at T_{PM}, the failed components of type ℓ, $\ell = 1,2,...,n$, are replaced by new and identical ones at a cost $c_0^{(\ell)}$ and a

PM action is performed on the whole system at a cost c_t. Notice that in the later case, some components may have already failed during the operation in the time interval $(0, T_{PM})$ and thus have been unemployed.

In order to assess the cost rate for the described maintenance strategy, we first compute the expected cost during a maintenance cycle. If the system is alive at the planned time T_{PM}, then the expected maintenance cost will be

$$\bar{H}(T_{PM})\left\{c_t + \sum_{\ell=1}^{L} c_0^{(\ell)} E[N^{(\ell)}(T_{PM}) \mid T > T_{PM}]\right\},$$

where the conditional expectation can be obtained as

$$\begin{aligned} E[N^{(\ell)}(T_{PM}) \mid T_{PM} < T] = \frac{1}{\bar{H}(T_{PM})} & \sum_{i_1=1}^{n_1} \cdots \sum_{i_L=1}^{n_L} i_\ell \Psi(i_1, ..., i_L) \\ & \times \prod_{j \in E} \left\{ \sum_{k_j=0}^{n_j} \bar{P}_{n_j-k_j}^{(j)}(T_{PM}) \binom{n_j}{k_j} F_j^{k_j}(T_{PM}) \bar{F}_j^{n_j-k_j}(T_{PM}) \right\} \\ & \times \prod_{j \in E^c} \left\{ \sum_{k_j=0}^{i_j} P_{i_j-k_j}^{(j)}(T_{PM}) \binom{n_j}{k_j} F_j^{k_j}(T_{PM}) \bar{F}_j^{n_j-k_j}(T_{PM}) \right\} \end{aligned}$$

and $\bar{H}(T_{PM})$ is defined in (2). On the other hand, if the system fails before the age of the system reaches scheduled time T_{PM}, then the expected cost is $c_r H(T_{PM})$.

Summing up, the expected cost during one maintenance cycle can be obtained as

$$\begin{aligned} \mathscr{C}(T_{PM}) = c_t + (c_r - c_t) H(T_{PM}) & \\ + \sum_{\ell=1}^{L} c_0^{(\ell)} & \sum_{i_1=1}^{n_1} \cdots \sum_{i_L=1}^{n_L} i_\ell \Psi(i_1, ..., i_L) \\ \times & \prod_{j \in E} \left\{ \sum_{k_j=0}^{n_j} \bar{P}_{n_j-k_j}^{(j)}(T_{PM}) \binom{n_j}{k_j} F_j^{k_j}(T_{PM}) \bar{F}_j^{n_j-k_j}(T_{PM}) \right\} \\ \times & \prod_{j \in E^c} \left\{ \sum_{k_j=0}^{i_j} P_{i_j-k_j}^{(j)}(T_{PM}) \binom{n_j}{k_j} F_j^{k_j}(T_{PM}) \bar{F}_j^{n_j-k_j}(T_{PM}) \right\}. \end{aligned}$$

Therefore, the expected cost rate can be expressed as

$$\eta(T_{PM}) = \frac{\mathscr{C}(T_{PM})}{E[\min(T, T_{PM})]},$$

where

$$E[\min(T, T_{PM})] = \sum_{i_1=0}^{n_1} \cdots \sum_{i_L=0}^{n_L} \Psi(i_1, \ldots, i_L) \int_0^{T_{PM}} P\left(N^{(1)}(t) = i_1, \ldots, N^{(L)}(t) = i_L\right) dt,$$

and $P\left(N^{(1)}(t) = i_1, \ldots, N^{(L)}(t) = i_L\right)$ is given in Proposition 2.1. The aim is to find the amount of T_{PM}^* which minimizes the cost function $\eta(T_{PM})$.

To illustrate the application of the PM model and the methodology developed above, in the following a numerical example is conducted.

Example 3.1. The reliability of the ABS is evaluated in Example 2.2. Recall that it contains 10 components of four types M, H, C and P. The lifetime distributions and shock processes for each type of components are summarized in Table 1. In this example, we apply a T_{PM}-policy on the ABS. Table 2 that contains the optimum PM time T_{PM}^* and the optimum cost $\eta(T_{PM}^*)$, shows the impact of different cost parameters c_t and c_r on the optimal time T_{PM}^*. As it is observed, when c_t increases, then both T_{PM}^* and the optimum cost $\eta(T_{PM}^*)$ increases. Note that an increase in c_t leads to the decision to perform preventive repair later. Also, note that when c_r increases, then, as expected, the PM time declines and hence an earlier PM action will be carried out.

Table 2: The optimal PM time T_{PM}^* and $\eta(T_{PM}^*)$ for different values of c_r and c_t with $c_0 = (10, 20, 15, 12)$.

	c_t					
	70		100		130	
c_r	T_{PM}^*	$\eta(T_{PM}^*)$	T_{PM}^*	$\eta(T_{PM}^*)$	T_{PM}^*	$\eta(T_{PM}^*)$
250	0.837	163.246	1.006	190.235	1.289	207.738
300	0.773	172.384	0.899	203.785	1.035	228.479
350	0.728	180.200	0.836	214.682	0.939	243.103

The cost functions are plotted in Figure 3(a) for different values of c_t and fixed values of $c_r = 250$ and $c_0 = (10, 20, 15, 12)$. Also, the cost functions are plotted in Figure 3(b) for different values of c_r and the fixed value $c_t = 130$.

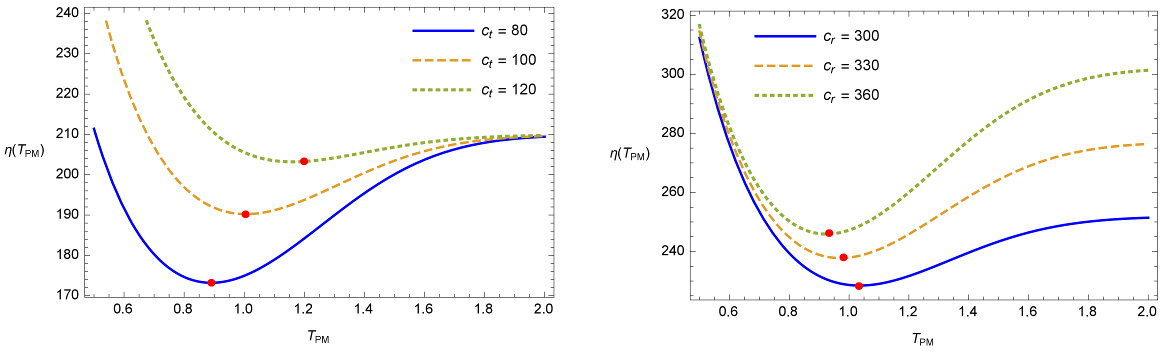

Figure 3: (a) The cost function of the ABS for different c_t. (b) The cost function of the ABS for different c_r.

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New Expressions for the Variance of Transformed Random Variables

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Abstract: The concept of mean residual lifetime and mean inactivity time play crucial roles in reliability, risk theory and life testing. In this regard, we introduce weighted mean residual lifetime and mean inactivity time functions by considering a non-negative weight function. Based on this function, we provide expressions for the variance of transformed random variable in terms of the square of weighted mean residual lifetime and mean inactivity time functions.

Keywords: Mean inactivity time function, Mean residual lifetime function, Variance.

1 Introduction

Let X be a non-negative absolutely continuous random variable denoting the lifetime of a system or a component or a living organism with cumulative distribution function (CDF) $F(x) = \mathbb{P}(X \leq x)$ and probability density function (PDF) $f(x)$. If X denotes the lifetime a system or a component under the condition that the system has survived up to age t, then the residual lifetime is defined by $X_t = [X - t \,|\, X > t]$, where as usual $[X|B]$ denotes a random variable having the same distribution of X conditioned on B. The *mean residual life* (MRL) function of X is defined as

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$$m(t) = \mathbb{E}[X - t | X > t] = \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x)\,\mathrm{d}x, \tag{1}$$

for all $t \geq 0$ such that $\overline{F}(t) > 0$. Analogously, under the condition that the system has been found failed before time t, the inactivity time is defined by $X_{[t]} = [t - X \,|\, X \leq t]$. In fact, $X_{[t]}$ denotes a random variable whose distribution is the same as the conditional distribution of $t - X$ given that $X \leq t$. It is worth emphasizing that in many realistic situations, the random lifetime can refer to the past. For instance, consider a system whose state is observed only at certain preassigned inspection times. If at time t, the system is inspected for the first time and it is found to be "down", then the failure relies on the past (see e.g. Kayid and Ahmad [12] and Di Crescenzo and Longobardi [5]). Now, we recall the MIT function of X which is defined by

$$\widetilde{\mu}(t) = \mathbb{E}[t - X | X \leq t] = \frac{1}{F(t)} \int_0^t F(x)\,\mathrm{d}x, \qquad t \in D, \tag{2}$$

where $D := \{t > 0 : F(t) > 0\}$ and where $\mathbb{E}[\cdot]$ means expectation. Assuming that $\widetilde{\mu}(t)$ is a differentiable function, from (2) we get

$$\widetilde{\mu}'(t) = 1 - \tau(t)\widetilde{\mu}(t), \qquad t \in D, \tag{3}$$

where

$$\tau(x) = \frac{f(x)}{F(x)}, \qquad x \in D, \tag{4}$$

denotes the reversed hazard rate function of X.

The mean residual life function has been employed in life lengths studies by various authors, see e.g. Hall and Wellner [7], Bhattacharjee [4] and Hollander and Proschan [8] and the references therein. Kayid and Ahmad [12] (see also Ahmad et al. [1]) studied stochastic comparisons based on the MIT function under the reliability operations of convolution and mixture. Badia and Berrade [3] gave an insight into properties of the MIT in mixtures of distributions. Some further properties of MIT function are widely studied and investigated in Finkelstein [10] and Kundu *et al.* [13]

and the references therein. Moreover, Izadkhah and Kayid [11] used the harmonic mean average of the MIT function to propose a new stochastic order. Recently, Toomaj and Di Crescenzo [14] showed that the variance of a random variable can be represented in terms of the square of the MRL and MIT functions. Therefore, it is not surprising that the MIT function has been the object of several investigations. The aim of the present paper is to define a new version of MRL and MIT functions, namely the weighted MRL and MIT functions and to show some applications of such measures.

2 Main results

The aim of this section is to investigate on the weighted mean inactivity time function by applying *the cumulative weight function*, say. For this aim, we consider a non-negative and differentiable function $\phi(x)$ in $[0,\infty)$. The cumulative weight function is defined as

$$\psi(x) := \int_0^x \phi(u)\,\mathrm{d}u, \qquad x \geq 0. \tag{5}$$

This function plays a pivotal role in achieving our results. Specifically, given the random lifetime X, we analyze various properties of the transformed random variable $\psi(X)$, where the latter may be viewed as an increasing time-change of X. Let $\overline{F}(t) = 1 - F(t)$ be the survival function of X, and let

$$\lambda(x) = \frac{f(x)}{\overline{F}(x)}, \qquad \forall\, x \geq 0 : \overline{F}(x) > 0 \tag{6}$$

denote the hazard rate function of X. For example, if we consider $\phi(x) = \lambda(x)$, then (5) gives the cumulative hazard function of X. Due to (5), it is clear that $\psi(x)$ is an increasing function of $x > 0$ such that $\psi(0) = 0$, since $\psi'(x) = \phi(x) \geq 0$. Additionally, if the weight function $\phi(x)$ is increasing (decreasing) in $x > 0$, then $\psi(x)$ is convex (concave). Now, we introduce the *weighted mean residual life* (WMRL) function given by

$$m_\psi(t) = m_{\psi(X)}(t) = \mathbb{E}[\psi(X) - \psi(t)|X > t] = \frac{1}{\overline{F}(t)} \int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x, \tag{7}$$

for all $t \geq 0$ such that $\overline{F}(t) > 0$. In particular, when $\psi(t) = t$, and hence $\phi(t) = 1$, then Eq. (7) coincides with the MRL function (1). Hereafter, we focus on the two nonparametric classes of lifetime distributions based on increasing and decreasing nature of weighted mean residual life function $m_\psi(t)$.

Definition 2.1. A non-negative random variable X is said to have increasing (decreasing) weighted mean residual life function, denoted by IWMRL (DWMRL), if $m_\psi(t)$ is an increasing (decreasing) function of $t \geq 0$.

Hereafter, we provide different conditions such that $m_\psi(x)$ is monotonic. To this aim, we recall that X is said to be increasing (decreasing) in mean residual life, i.e. IMRL (DMRL), if the MRL function $m(x)$ is increasing (decreasing) in x.

Theorem 2.2. *Let X be an absolutely continuous non-negative random variable. If $\phi(x)$ is increasing (decreasing) in x, and if X is IMRL (DMRL), then X is $IWMRL$ $(DWMRL)$.*

Now, let us consider the following example.

Example 2.3. Let us consider $\phi(t) = \Lambda(t) = -\log \overline{F}(t)$, and thus $\psi(t) = \int_0^t \Lambda(\tau)\,\mathrm{d}\tau$. In this case, from (7) we have

$$m_\psi(t) = -\frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x) \log \overline{F}(x)\,\mathrm{d}x, \qquad t > 0.$$

Hence, making use of Eq. (14) of Asadi and Zohrevand [2], we have

$$m_\psi(t) = \mathscr{E}(X;t) - m(t) \log \overline{F}(t), \qquad t > 0, \tag{8}$$

where

$$\mathscr{E}(X;t) = -\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)} \log \frac{\overline{F}(x)}{\overline{F}(t)}\,\mathrm{d}x, \qquad t > 0,$$

is the dynamic cumulative residual entropy (DCRE) of X. Recalling Corollary 4.4 of Asadi and Zohrevand [2], we have that if X is IMRL, then $\mathscr{E}(X;t)$ is increasing in t, and thus from (8) we obtain that in this case X is IWMRL. This conclusion can also be obtained from Theorem 2.2.

Hereafter, in the main result of this section, we express the variance of $\psi(X)$ in terms of the WMRL function (7) (see Toomaj and Di Crescenzo [15]).

Theorem 2.4. *Let X be an absolutely continuous non-negative random variable. If the weighted mean residual life function (7) has finite second moment, i.e. $\mathbb{E}[m_\psi^2(X)] < \infty$, then*

$$\sigma^2[\psi(X)] = \mathbb{E}[m_\psi^2(X)]. \tag{9}$$

In the following theorem, by making use of the above results we first investigate the impact of the transformation $\psi(x)$ on the variance of a random variable.

Theorem 2.5. *Under the condition of Theorem 2.4, if the function $\phi(\cdot)$ is increasing (decreasing), such that $\phi(x) \geq 1$ $(0 \leq \phi(x) \leq 1)$ for all x in the support of X, then*

$$\sigma^2[\psi(X)] \geq (\leq)\, \sigma^2(X).$$

As an application of (9), let us consider the following example.

Example 2.6. Let $X_{1:m} = \min\{X_1, \ldots, X_m\}$ denote the minimum of independent and identically distributed absolutely continuous non-negative random variables $X_1, \ldots, X_m$ coming from CDF $F(x)$. Denote by $\overline{F}_{1:m}(x) = \mathbb{P}(X_{1:m} > x) = [\overline{F}(x)]^m$, $x \geq 0$, the survival function of $X_{1:m}$. Hence, by setting $\psi(t) = F(t)$, and thus $\phi(t) = f(t)$, from (7) we obtain, for $t > 0$,

$$m_{\psi(X_{1:m})}(t) = \frac{1}{\overline{F}_{1:m}(t)} \int_t^\infty f(x)\overline{F}_{1:m}(x)\,\mathrm{d}x = \frac{1}{[\overline{F}(t)]^m} \int_t^\infty f(x)[\overline{F}(x)]^m\,\mathrm{d}x = \frac{\overline{F}(t)}{m+1}.$$

By making use of Theorem 2.4, thus the variance of the probability integral transformation $F(X_{1:m})$ can be obtained as

$$\sigma^2[F(X_{1:m})] = m \int_0^\infty f(x)[\overline{F}(x)]^{m-1} \left[\frac{\overline{F}(x)}{m+1}\right]^2 \mathrm{d}x = \frac{m}{(m+1)^2(m+2)}.$$

In analogy with (7), we introduce the *weighted mean inactivity time* (WMIT) function as

$$\widetilde{\mu}_{\psi}(t)=\widetilde{\mu}_{\psi(X)}(t)=\mathbb{E}[\psi(t)-\psi(X)|X\leq t]=\frac{1}{F(t)}\int_0^t \phi(x)F(x)\,\mathrm{d}x, \qquad t\in D \tag{10}$$

In particular, when $\psi(t)=t$, and hence $\phi(t)=1$, then Eq. (10) coincides with the MIT function (2). In what follows, we implicitly assume that

$$\mathbb{E}[\psi(X)]=\int_0^{\infty}\psi(x)f(x)\,\mathrm{d}x<\infty, \tag{11}$$

to ensure the finiteness of $\widetilde{\mu}_{\psi}(t)$. Henceforward, we investigate some properties of the WMIT function given in (10). Now, consider the following definition.

Definition 2.7. A non-negative random variable X is said to have increasing weighted mean inactivity time function, denoted by IWMIT, if $\widetilde{\mu}_{\psi}(t)$ is an increasing function of $t\in D$.

As an application of (10), let us consider the following example.

Example 2.8. Let us assume that $\phi(t)=\tau(t)\widetilde{\mu}(t)=1-\widetilde{\mu}'(t)$, where the last equality is due to (3). From (5) we thus have $\psi(t)=\int_0^t\phi(u)\,\mathrm{d}u=t-\widetilde{\mu}(t)$ for all $t>0$. In this case, from (10) we get

$$\widetilde{\mu}_{\psi}(t)=\frac{1}{F(t)}\int_0^t f(x)\widetilde{\mu}(x)\,\mathrm{d}x, \qquad t\in D.$$

Hence, making use of Theorem 5.2. of Di Crescenzo and Longobardi [5], we have

$$\widetilde{\mu}_{\psi}(t)=\mathscr{CE}(X;t), \qquad t>0, \tag{12}$$

where $\mathscr{CE}(X;t)$ is known as the *dynamic cumulative entropy* of X. Recalling Corollary 6.1 of Di Crescenzo and Longobardi [5], we have that if X is IMIT, then $\mathscr{CE}(X;t)$ is increasing in t, and thus from (12) we obtain that X is IWMIT in this case.

The following result deals with the WMIT and MIT functions.

Lemma 2.9. *Let X be an absolutely continuous non-negative random variable with weighted mean inactivity time function $\widetilde{\mu}_\psi(t)$ defined as in (10). Assume that there exist non-negative constants m and M such that $m \le \phi(t) \le M$ for all $t \ge 0$. Then*

$$m \le \frac{\widetilde{\mu}_\psi(t)}{\widetilde{\mu}(t)} \le M \qquad \textit{for all } t \in D. \tag{13}$$

Lemma 2.9 allows us to obtain ordering relations between the WMIT and MIT functions. Indeed, (i) if $M = 1$, then $\widetilde{\mu}_\psi(t) \le \widetilde{\mu}(t)$ for all $t \in D$; (ii) if $m = 1$, then $\widetilde{\mu}_\psi(t) \ge \widetilde{\mu}(t)$ for all $t \in D$.

For instance, if $\phi(t) = \overline{F}(t)$, then M=1, and $\widetilde{\mu}_\psi(\infty) = \mathbb{E}[|X - X'|]/2$, where X' is an independent copy of X, provided that the condition (11) is satisfied.

Recently, Toomaj and Di Crescenzo [14] showed that the variance of a random variable X can be represented in terms of MIT function as follows:

$$\sigma^2(X) = \mathbb{E}[\widetilde{\mu}^2(X)], \tag{14}$$

provided that the expectation exists. In what follows, we extend the result (14) to the case of the transformed random variable $\psi(X)$, where $\psi(x)$ is the cumulative weight function defined in (5). Indeed, in the following theorem we express the variance of $\psi(X)$ in terms of the WMIT function (10) (see Di Crescenzo and Toomaj [16]).

Theorem 2.10. *Let X be an absolutely continuous non-negative random variable with WMIT function $\widetilde{\mu}_\psi(t)$, and having finite second moment $\mathbb{E}[\psi^2(X)]$. Then*

$$\sigma^2[\psi(X)] = \mathbb{E}[\widetilde{\mu}_\psi^2(X)]. \tag{15}$$

As an application of Eq. (15), let us consider the following example.

Example 2.11. Consider a parallel system composed by m units having lifetimes $X_1, \ldots, X_m$, which are independent and identically distributed (i.i.d.) absolutely continuous random variables with CDF $F(x)$. The system lifetime is thus $X_{m:m} = \max\{X_1, \ldots, X_m\}$, whose CDF is given by

$F_{m:m}(x) := \mathbb{P}(X_{m:m} \le x) = [F(x)]^m$, $x \ge 0$. Setting $\psi(t) = F(t)$, and thus $\phi(t) = f(t)$, from (10) we obtain, for $t > 0$,

$$\widetilde{\mu}_{\psi(X_{m:m})}(t) = \frac{1}{F_{m:m}(t)} \int_0^t f(x) F_{m:m}(x)\,\mathrm{d}x = \frac{1}{[F(t)]^m} \int_0^t f(x)[F(x)]^m\,\mathrm{d}x = \frac{F(t)}{m+1}.$$

Thanks to the use of Eq. (15), thus the variance of the probability integral transformation $F(X_{m:m})$ can be obtained as

$$\sigma^2[F(X_{m:m})] = m \int_0^\infty f(x)[F(x)]^{m-1} \left[\frac{F(x)}{m+1}\right]^2 \mathrm{d}x = \frac{m}{(m+1)^2(m+2)}.$$

In the next theorem, we state that when the weight function is bounded between two real numbers, the ratio of standard deviation of transformed random variable with respect to the standard deviation of the associated random variable also lies down between the same bounds.

Theorem 2.12. *Under the conditions of Lemma 2.9, it holds that*

$$m \le \frac{\sigma[\psi(X)]}{\sigma(X)} \le M.$$

In particular, (i) if $m = 0$ and $M = 1$, then $\sigma[\psi(X)] \le \sigma(X)$ and, (ii) if $m = 1$ and $M < \infty$, then $\sigma[\psi(X)] \ge \sigma(X)$.

Proof. The proof is immediately obtained from (13) and recalling (14) and (15). □

Now, let us consider two applications in the following examples.

Example 2.13. Let X and Y be non-negative random lifetimes with CDFs F and G, respectively. Consider the function $\psi(t) = G^{-1}F(t)$, which is increasing in $t > 0$. Due to (5), we have that $\phi(x) \ge 1$ if and only if $\psi(t) - t$ is increasing in t. Supposing that $X \le_{disp} Y$, (we recall that X is said to be smaller than Y in the dispersive order, denoted by $X \le_{disp} Y$, if and only if,

$$G^{-1}(F(t)) - t \qquad \text{is increasing in } t > 0, \tag{16}$$

where $G^{-1}(u) = \inf\{x \in \mathbb{R}^+ : G(x) \ge u\}$, $u \in [0,1]$, denotes the left-continuous quantile function of $G(x)$.) One can conclude that $\psi(t) - t =$

$G^{-1}F(t)-t$ is increasing t by recalling (16). Making use of Theorem 2.12, we have

$$\sigma[\psi(X)] = \sigma(G^{-1}F(X)) \geq \sigma(X).$$

By noting that $G^{-1}F(X) \stackrel{d}{=} Y$, where $\stackrel{d}{=}$ means equality in distribution, we obtain the well-known result $\sigma(X) \leq \sigma(Y)$.

Example 2.14. Assume that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random lifetimes with the common CDF $F(x)$ and PDF $f(x)$. The ith smallest value is usually called the ith order statistic, and is denoted by $X_{i:n}$, $i = 1, 2, \ldots, n$. Let us set $\psi(x) = F(x)$ and thus $\phi(x) = f(x)$. If S is the support of f, then

$$\inf_{x\in S} f(x) =: m \leq f(x) \leq M := \sup_{x\in S} f(x).$$

It is known that the probability integral transform $V_i = F(X_{i:n})$ has a beta distribution with parameters i and $n-i+1$, respectively. Since

$$\sigma^2[V_i] = \sigma^2[F(X_{i:n})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}, \qquad i = 1, 2, \ldots, n,$$

from Theorem 2.12 we have

$$\frac{i(n-i+1)}{M^2(n+1)^2(n+2)} \leq \sigma^2[X_{i:n}] \leq \frac{i(n-i+1)}{m^2(n+1)^2(n+2)}, \qquad i = 1, 2, \ldots, n$$

provided that $0 < m \leq M < \infty$. Specifically, after some simplifications the average variance of the order statistics is bounded as follows:

$$\frac{1}{6M^2(n+1)} \leq \frac{1}{n}\sum_{i=1}^{n} \sigma^2[X_{i:n}] \leq \frac{1}{6m^2(n+1)}.$$

The latter result is useful to show that when n goes to infinity, then the average variance of the order statistics vanishes, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sigma^2[X_{i:n}] = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sigma^2[X_i] = 0,$$

provided that $0 < m \leq M < \infty$.

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