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A TWO-PARAMETER DISTRIBUTION BY MIXING WEIBULL AND LINDLEY MODELS

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Abstract. In this paper, we introduce a new lifetime distribution by mixing the Weibull and Lindely distributions. We assume that the scale parameter of the Weibull distribution is a random variable having the Lindely distribution. The shapes of the density and hazard rate functions are discussed. Further, some properties of the distribution are obtained, involving quantiles and moments. The distribution parameters are estimated by maximum likelihood method and its performance is evaluated by a simulation study. Applicability of the distribution among other competitive distributions is illustrated by fitting a practical data set and using some goodness-of-fit statistics.

1. Introduction

In several practical situations, objects in a certain population differ substantially from each other, hence the heterogeneity of such objects should be considered for accurate data analysis of this population. Therefore, mixture distribution is a recommended model for analyzing the heterogeneity. Another issue must be taken into account is the different nature of the practical data which requires introducing new distributions with various hazard rate (hr) shapes to model and analyze such data.

The two aims above are investigated by introducing a new mixture distribution, named Weibull Lindley distribution via mixing the Lindely and Weibull distributions in a different manner than used in Asgharzadeh et al. [2]. Also, the new distribution has decreasing and unimodal hazard rates shapes and its construction

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as follows.
Let $X \mid \lambda$ follow the Weibull distribution with the probability density function (pdf)

$$f(x \mid \lambda) = \lambda \alpha x^{\alpha-1} e^{-\lambda x^{\alpha}}, \quad x > 0; \quad \alpha, \lambda > 0,$$

and $\lambda \mid \beta$ follows the Lindley distribution (Lindley [7]) with the pdf

$$f(\lambda \mid \beta) = \frac{\beta^2}{1+\beta}(1+\lambda)e^{-\beta\lambda}, \quad \lambda > 0; \quad \beta > 0.$$

Hence, the marginal distribution of X is called the Weibull-Lindley (WeL) distribution. The pdf of X is obtained as

$$f(x) = \frac{\alpha\beta^2\, x^{\alpha-1}}{1+\beta} \int_0^{\infty} \lambda(1+\lambda)e^{-(\beta+x^{\alpha})\lambda}\, \mathrm{d}\lambda,$$

and after some algebra, we gets the WeL pdf as

$$f(x) = \frac{\alpha\beta^2 x^{\alpha-1}}{1+\beta} \frac{2+\beta+x^{\alpha}}{(\beta+x^{\alpha})^3}, \quad x > 0; \quad \alpha, \beta > 0. \tag{1.1}$$

Moreover, the cumulative distribution function (cdf) of the WeL distribution is

$$F(x) = 1 - \frac{\beta^2}{1+\beta} \frac{1+\beta+x^{\alpha}}{(\beta+x^{\alpha})^2}, \tag{1.2}$$

hence, the corresponding reliability (survival) function is given by

$$R(x) = \frac{\beta^2}{1+\beta} \frac{1+\beta+x^{\alpha}}{(\beta+x^{\alpha})^2}. \tag{1.3}$$

In reliability analysis, usefulness of the model (1.1) comes in noting that X can be the lifetime of a component and λ is the scale parameter of its distribution. If the population having some variability in its scale parameter, then this variability can be explained by the distribution for λ. Moreover, comparing the WeL distribution with Weibull and Lindley distributions implies the flexibility of WeL in terms of its hazard rate shapes as shall be shown later. Also, we shall see later that it has decreasing and unimodal (upside-down bathtub) hazard rates. Decreasing and unimodal shaped hazard rates have many applications in reliability and survival analysis. It may be difficult to know why the lifetime of an object having a decreasing hazard rate. However, it would seem to correspond to some physical mechanisms of improvement with the time. In reliability, this may happen in situations where the product manufacturer continues to improve in-serve product by implementing corrective actions. On the other side, as mentioned by Lai and Xie [6], when the main reasons of the failures of products are caused by fatigue and corrosion, the failure rates of those products will exhibit unimodal shapes. Further, in some medical situations, such as breast cancer and infection with some new viruses, the hazard rate has a unimodal shape, see Demicheli et al. [4]. Another example in epidemiology is that the patients with tuberculosis have a risk which initially increases and then decreases after the treatment. The Weibull Lindley distribution proposed by Asgharzadeh et al [2] does not allow a unimodal hazard rate shape. So, this distribution is not suitable for modeling data with unimodal hazard rates.

2. Shape characteristics

In this section, we discuss the shape characteristics of the pdf, hrf and rhrf of the WeL distribution.

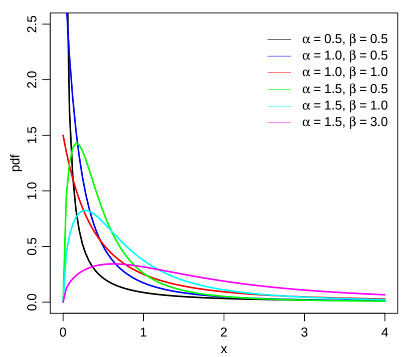

FIGURE 1. Plots of the WeL density for some parameter values.

2.1. **Shape of pdf.** we can see from (1.1) that

$$\lim_{x\to 0} f(x) = \begin{cases} \infty & \alpha < 1 \\ \frac{2+\beta}{\beta(1+\beta)} & \alpha = 1 \\ 0 & \alpha > 1, \end{cases}$$

and $\lim_{x\to\infty} f(x) = 0$. Figure 1 shows the pdf of the WeL distribution for some selected choices of α and β. From it, we see that the pdf of WeL distribution is decreasing for $\alpha \le 1$ and unimodal for $\alpha > 1$. Features of the pdf of WeL distribution are discussed theoretically in the next theorem.

Theorem 2.1. *The pdf of WeL distribution given by* (1.1) *is decreasing for* $\alpha \le 1$ *and unimodal for* $\alpha > 1$.

Proof. The logarithm of (1.1) is

$$\ln f(x) = Constant + (\alpha - 1)\ln x + \ln(2 + \beta + x^{\alpha}) - 3\ln(\beta + x^{\alpha}).$$

We have

$$\begin{aligned} \frac{d}{dx}\log f(x) &= \frac{\alpha - 1}{x} + \frac{\alpha x^{\alpha-1}}{2+\beta+x^{\alpha}} - \frac{3\,\alpha\, x^{\alpha-1}}{\beta + x^{\alpha}} \\ &= \frac{\alpha - 1}{x} - \frac{2\alpha\, x^{\alpha-1}(3+\beta+x^{\alpha})}{(\beta+x^{\alpha})(2+\beta+x^{\alpha})}. \end{aligned}$$

If $\alpha \le 1,$ we easily see that

$$\frac{d}{dx}\log f(x) < 0.$$

Hence, $f(x)$ is decreasing for all x. For $\alpha > 1$, $\frac{d}{dx}\log f(x)$ has a global maximum at some point x_0, where x_0 is the root of the equation $\frac{d}{dx}\log f(x) = 0$.

2.2. **Hazard rate shape.**
The hazard rate function (hrf) corresponding to (1.1) and (1.2) is given by

$$h(x) = \frac{\alpha x^{\alpha-1}\left(2+\beta+x^{\alpha}\right)}{\left(\beta+x^{\alpha}\right)\left(1+\beta+x^{\alpha}\right)}. \tag{2.1}$$

The behavior of $h(x)$ when $x \to 0$ and $x \to \infty$, respectively, are given by

$$\lim_{x\to 0} h(x) = \begin{cases} \infty & \alpha < 1 \\ \frac{(2+\beta)}{\beta(1+\beta)} & \alpha = 1 \\ 0 & \alpha > 1 \end{cases} \quad \textit{and} \quad \lim_{x\to\infty} h(x) = 0.$$

Figure 2, show the hrf $h(x)$ of the WeL distribution for some choices of α and β.

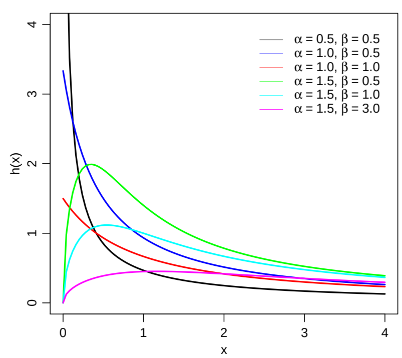

FIGURE 2. Plots of the WeL hrf for some parameter values.

The next theorem investigates the shapes for the hazard rate function of the WeL distribution.

Theorem 2.2. *The hazard rate function of the WeL distribution in* (2.1) *is decreasing for* $\alpha \le 1$ *and unimodal for* $\alpha > 1$.

Proof. For the pdf (1.1), we have

$$\eta(x) = -\frac{f'(x)}{f(x)} = -\frac{\alpha-1}{x} + \frac{2\alpha\, x^{\alpha-1}(3+\beta+x^{\alpha})}{(\beta+x^{\alpha})(2+\beta+x^{\alpha})},$$

hence, we have

$$\eta'(x) = \frac{\alpha-1}{x^2} + \frac{x^{2\alpha-2}\Big[2\beta(2\alpha-5)+\beta^2(\alpha-3)-6\Big] + x^{3\alpha-2}\Big[\beta(\alpha-3)-\alpha-7\Big] - x^{4\alpha-2}}{(\beta+x^{\alpha})^2(2+\beta+x^{\alpha})^2}.$$

For $\alpha \le 1$, we have $\eta'(x) < 0$. For $\alpha > 1$, $\eta(x)$ has a global maximum at some point x_0, where x_0 is the root of the equation $\eta'(x) = 0$. Therefore, this part follows from Glasers Theorem (Glaser [8]).

3. Some properties of the WeL distribution

In this section, we obtain some properties of the WeL distribution, involving quantiles and moments.

3.1. **Quantiles and moments.** For the WeL distribution, the pth quantile x_p is the solution of $F(x_p) = p$, hence

$$x_p = \left((1+\beta)\left((1+\frac{x_p^\alpha}{\beta})^2(1-p)-1\right)\right)^{1/\alpha},$$

which is the base of generating WeL random variates.

Now, we obtain moments of the WeL distribution. If $X \sim WeL(\alpha, \beta)$, then it can be shown that $E(X) = \infty$, and generally $E(X^r) = \infty, \quad r \geq 1$, hence, all moments of the distribution are infinite. Therefore, the WeL distribution has no mean. Thus, in practice, if $X_1, X_2, \cdots, X_n$ is a random sample drawn from the WeL distribution, then the mean $\bar{X}$ does not tend to a particular value. Due to this obstacle, we get another type of moments for this distribution. The negative moments are of interest in various applications in life testing and estimation purposes. Therefore, we get them for this distribution. The r^{th} negative moment of the WeL distribution is

$$\begin{aligned} E(X^{-r}) &= E(E(x^{-r} \mid \lambda)) = \Gamma(1-\frac{r}{\alpha})E(\lambda^{r/\alpha}) \\ &= \Gamma(1-\frac{r}{\alpha})\frac{(r/\alpha)!(\beta+\frac{r}{\alpha}+1)}{\beta^{r/\alpha}(\beta+1)}, \quad r<\alpha, \quad r=1,2,\cdots. \end{aligned}$$

3.2. **Stochastic ordering.** Comparative behavior of positive continuous random variables can be judged by stochastic ordering. Therefore, let us recall the next concepts.

A random variable X_1 is said to be smaller than a random variable X_2 in the

(i) stochastic order ($X_1 \prec_{st} X_2$) if $F_{X_1}(x) \geq F_{X_2}(x)$ for all x,

(ii) hazard rate order ($X_1 \prec_{hr} X_2$) if $h_{X_1}(x) \geq h_{X_2}(x)$ for all x,

(iii) likelihood ratio order ($X_1 \prec_{lr} X_2$) if $\frac{f_{X_1}(x)}{f_{X_2}(x)}$ decreases in x.

The likelihood ratio order implies hazard rate order which in turn implies stochastic order. The following theorem presents the stochastic ordering for the WeL distribution. The proof is easy and omitted.

Theorem 3.1. *Let $X_i \sim WeL(\alpha_i, \beta_i), i = 1, 2,$ be two random variables. If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 \leq \beta_2$, and if $\beta_1 = \beta_2 = \beta \geq 1$ and $\alpha_1 \leq \alpha_2$, then $X_1 \prec_{lr} X_2 \Rightarrow X_1 \prec_{hr} X_2 \Rightarrow X_1 \prec_{st} X_2$.*

4. Maximum Likelihood Estimation

Let $x_1, x_2, \cdots, x_n$ be the observed values of a random sample taken from the $WeL(\alpha, \beta)$ distribution, then the log-likelihood function is

$$\begin{aligned} \ln L(\alpha,\beta) = n \ln\alpha + 2n \ln\beta - n \ln(1+\beta) + \sum_{i=1}^{n} \ln(2+\beta+x_i^{\alpha}) \\ + (\alpha-1)\sum_{i=1}^{n} \ln x_i - 3\sum_{i=1}^{n} \ln(\beta+x_i^{\alpha}). \end{aligned} \tag{4.1}$$

The maximum likelihood estimators (MLEs) of α and β, say $\hat{\alpha}$ and $\hat{\beta}$, are the solutions of the equations

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha} \ln x_i}{2+\beta+x_i^{\alpha}} + \sum_{i=1}^{n} \ln x_i - 3\sum_{i=1}^{n} \frac{x_i^{\alpha} \ln x_i}{\beta+x_i^{\alpha}} = 0,$$

and

$$\frac{2n}{\beta} - \frac{n}{1+\beta} + \sum_{i=1}^{n} \frac{1}{2+\beta+x_i^{\alpha}} - 3\sum_{i=1}^{n} \frac{1}{\beta+x_i^{\alpha}} = 0.$$

5. Monte Carlo simulation study

In this section, we assess the performance of the MLE's of the parameters with respect to sample size n for the $WeL(\alpha, \beta)$ distribution. The assessment of performance is based on a simulation study by using the Monte Carlo method. Let $\hat{\alpha}$ and $\hat{\beta}$ be the MLEs of the parameters α and β, respectively. We compute the mean square error (MSE) and bias of the MLEs of the parameters α and β based on the simulation results of $N = 2000$ independence replications. The results are summarized in Table 1 for different values of n, α and β. From Table 1, the results verify that MSE and bias of the MLEs of the parameters decrease with respect to sample size n increases. Hence, we can see the MLEs of α and β are consistent estimators.

Table 1. MSEs and Average biases(values in parentheses) of the simulated estimates.

		$\alpha = 0.5$	$\beta = 0.5$	$\alpha = 1.0$	$\beta = 1.5$
n	30	0.038 (0.162)	1.038 (0.939)	0.025 (0.004)	0.714 (-0.830)
	50	0.028 (0.146)	0.880 (0.896)	0.013 (-0.013)	0.709 (-0.833)
	100	0.023 (0.141)	0.822 (0.885)	0.006 (-0.030)	0.704 (-0.835)
	200	0.020 (0.137)	0.800 (0.884)	0.004 (-0.032)	0.700 (-0.835)

		$\alpha = 0.5$	$\beta = 1.0$	$\alpha = 1.5$	$\beta = 0.5$
n	30	0.029 (0.138)	0.055 (-0.091)	0.072 (-0.148)	6.352 (2.290)
	50	0.022 (0.128)	0.036 (-0.097)	0.058 (-0.177)	5.195 (2.166)
	100	0.018 (0.123)	0.023 (-0.097)	0.050 (-0.193)	4.612 (2.098)
	200	0.015 (0.117)	0.016 (-0.103)	0.048 (-0.204)	4.348 (2.060)

		$\alpha = 1.0$	$\beta = 0.5$	$\alpha = 1.5$	$\beta = 1.0$
n	30	0.042 (0.097)	2.982 (1.591)	0.101 (-0.247)	0.081 (0.075)
	50	0.023 (0.078)	2.716 (1.568)	0.098 (-0.277)	0.046 (0.075)
	100	0.013 (0.064)	2.414 (1.521)	0.093 (-0.288)	0.023 (0.064)
	200	0.007 (0.055)	2.243 (1.481)	0.093 (-0.297)	0.013 (0.062)

6. Practical data application

In this section, we present the application of the WeL model to an practical data set to illustrate its flexibility among a set of competitive models.

The data set is the Cancer Patients data. The data represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [5]. This data set consists of the observations:

Table 2. The data set

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98
6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28
9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14
79.05	1.35	2.87	5.62	7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13	1.76	3.25	4.50	6.25
8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76
12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69				

We compare the WeL model with a set of competitive models, namely Lindley distribution (Lindley [7]), Weibull Lindley distribution (WL) (Asgharzadeh et al., [2]), A new weighted Lindley distribution (NWL) (Asgharzadeh et al., [1]), The power Lindley distribution (PL) (Ghitany et al., [9]), The extended Lindley distribution (EL) (Bakouch et al., [3]), Weibull and Gamma distribution.

Table 3. Parameter estimates, standard errors, log-likelihood values and goodness of fit measures in data set.

Model	Parameter Estimation(s.e)	$-\log(L)$	K-S	p-value	AIC	BIC
$WeL(\alpha,\beta)$	$\hat{\alpha} = 1.7244\,(0.1281)$ $\hat{\beta} = 23.4951\,(6.3406)$	411.4565	0.039	0.9874	826.9131	832.6172
$Lindley(\beta)$	$\hat{\beta} = 0.1960\,(0.0123)$	419.5299	0.116	0.0623	841.0598	843.9118
$WL(\alpha,\lambda,\beta)$	$\hat{\alpha} = 1.0479 \times 10\,(0.0675)$ $\hat{\lambda} = 9.2678 \times 10^{-6}\,(0.0161)$ $\hat{\beta} = 1.0457 \times 10^{-1}\,(0.0093)$	414.0869	0.070	0.5555	834.1738	842.7298
$NWL(\alpha,\lambda)$	$\hat{\alpha} = 240.1998\,(588.0903)$ $\hat{\lambda} = 0.1961\,(0.0123)$	419.4645	0.116	0.0615	842.9289	848.633
$PL(\alpha,\beta)$	$\hat{\alpha} = 0.8303\,(0.0471)$ $\hat{\beta} = 0.2942\,(0.0369)$	413.3538	0.068	0.5889	830.7077	836.4117
$EL(\alpha,\lambda,\beta)$	$\hat{\alpha} = -2.0349\,(3.7241)$ $\hat{\lambda} = 0.0444\,(0.0521)$ $\hat{\beta} = 1.2240\,(0.2494)$	413.5721	0.088	0.2736	833.1442	841.7003
$Weibull(\alpha,\beta)$	$\hat{\alpha} = 1.0477\,(0.0675)$ $\hat{\beta} = 9.5600\,(0.8528)$	414.0869	0.069	0.5576	832.1738	837.8778
$Gamma(\alpha,\theta)$	$\hat{\alpha} = 1.1725\,(0.1308)$ $\hat{\theta} = 0.1252\,(0.0173)$	413.3678	0.073	0.4985	830.7356	836.4396

For each model, the MLEs and $-\log L = -\log L(\widehat{\alpha}, \widehat{\beta})$ values are computed. Consequently, the goodness-of-fit measures: Kolmogorov-Smirnov (K-S) statistics with their p-values, Akaike information criterion (AIC) and Bayesian information criterion (BIC) are evaluated. The required computations are carried out using the R software. The best model corresponds to lower $-\log L$, K-S, AIC and BIC values, and large p-values associated with K-S. Table 3 , for data set and each fitted model, list the MLEs of the parameters and their standard errors (in parentheses) for each model and the values of the goodness-of-fit measures.

The values of mentioned measures indicate that the WeL distribution is a strong competitor to other competitive distributions, moreover it is the best fit among others. To assess if the WeL distribution is appropriate, Figure 5 display the histogram of data set and the fitted density functions, and plots of the empirical and estimated cumulative distribution functions of these fitted distributions. From those graphical measures, we can conclude that the WeL distribution is a very suitable model to fit data set.

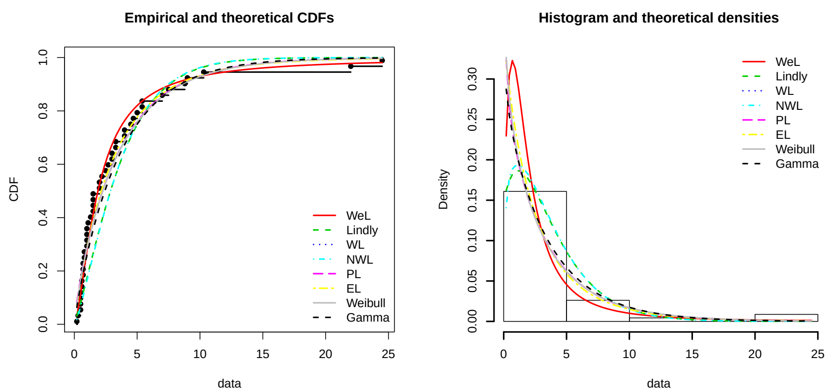

Figure 3. Estimated densities and Empirical and Estimated cdf for the Cancer Patients data set.

References

1. Asgharzadeh, A. Bakouch, H.S. Nadarajah, S. Sharafi, F. (2016), A new weighted Lindley distribution with application, *Brazilian Journal of Probability and Statistics*, **30** (1): 1-27.
2. Asgharzadeh, A. Nadarajah, S. Sharafi, F. (2018), Weibull Lindley Distribution, *REVSTAT*, **16**(1): 87-113.
3. Bakouch, H. S, Al-Zahrani, B. M. Al-Shomrani, A. A. Marchi, V. A. A. and Louzada, F. (2012), An extended Lindley distribution, *Journal of the Korean Statistical Society*, **41**: 7585. MR2933216
4. Demicheli, R. Bonadonna, G. Hrushesky, W. J. Retsky, M. W. Valagussa, P. (2004), Menopausal status dependence of the timing of breast cancer recurrence after surgical removal of the primary tumour, Breast Cancer Res **6**(6): 689-696.
5. Lee, E. T. and Wang, J. W. (2003), Statistical Methods for Survival Data Analysis, *Wiley, New York*, DOI: 10.1002/0471458546.
6. Lai, C. D. and Xie, M. (2006), Stochastic ageing and dependence for reliability. *New York: Springer*.
7. Lindley, D. V. (1958), Fiducial distributions and Bayes theorem, *Journal of the Royal Statistical Society Series*, B **20**: 102-107.
8. Glaser, R. E. (1980), Bathtub and related failure rate characterizations, *Journal of the American Statistical Association*, **75**: 667-672.
9. Ghitany, M. E. Al-Mutairi, D. K. Balakrishnan, N. and Al-Enezi, L. J. (2013), Power Lindley distribution and associated inference, *Computational Statistics and Data Analysis*, **64**: 20-33.